Steady, Free Circulation in a Stratified Quasi-Geostrophic Ocean

JOHN MARSHALL AND GEORGE NURSER

Atmospheric Physics Group, Department of Physics, Imperial College, London SW7 2BZ United Kingdom

(Manuscript received 5 August 1985, in final form 10 April 1986)

ABSTRACT

Steady solutions in which quasi-geostrophic potential vorticity is constant along a streamline of the flow are derived for a baroclinic ocean. Friction, transfer by geostrophic eddies, and wind forcing are treated as high-order effects that serve only to remove the indeterminacy of completely free flow. Solutions are obtained that are a generalization to a baroclinic ocean of Fofonoff's barotropic calculations. The vortex stretching permitted by stratification is found to allow gyres in which, in an integral sense, lateral down-gradient eddy transfer of potential vorticity, q, balances the wind-stress curl. Beneath the surface layer, the effect of eddies is then to make q uniform if q contours close on themselves.

Our simple solutions have many features in common with observations of the subtropical recirculation and with the mean flows obtained from eddy-resolving, quasi-geostrophic numerical models. In particular, the southern margin of the recirculation is found to recede progressively toward the line of zero wind-stress curl with increasing depth, the isopycnals sloping downward toward the northern boundary of the subtropical gyre.

1. Introduction

Recent mappings of the potential vorticity field, q, of the world's oceans (e.g., see Keffer, 1985; McDowell et al., 1982) show that there is strong advective control of the q contours. Rather than the q contours being coincident with latitude circles, as one would expect if the flow were weak, the q contours are strongly perturbed by the motion field, particularly in the "bowl" of the wind-driven circulation in the upper kilometer or so. Such evidence casts doubt on the relevance of linear theories of the mean circulation (where q is set by the planetary vorticity) and have encouraged us to reconsider the other asymptotic limit, that of steady conservative flow, where q is constant along streamlines of the flow ψ.

In this paper the consequences of viewing the wind-driven ocean circulation as an almost free, steady flow are examined. Forcing and dissipation act only to select the functional relationship between ψ and q. The constraint on the possible flow configurations that absolute vorticity must be constant along streamlines was considered by Fofonoff (1954), in the context of barotropic homogeneous ocean circulation theory. Niiler (1966) extended the study to investigate those forcing and dissipation processes that could equilibrate a Fofonoff gyre. He showed that a Fofonoff gyre, resonantly forced by a wind-stress curl, could not be equilibrated by transferring absolute vorticity laterally down the absolute vorticity gradient.

Here the studies of Fofonoff and Niiler are extended to consider steady, free circulation in a stratified ocean model governed by quasi-geostrophic dynamics. It is shown that by allowing vortex stretching to play its role in modifying the potential vorticity, it is possible to consider gyres in which, in an integral sense, the forcing due to the curl of the wind stress is balanced by lateral downgradient transfer of potential vorticity eddies. Beneath the directly wind-driven layer the effect of eddies is then to make q uniform, as in the model of Rhines and Young (1982). Our formulation, however, is complementary to theirs in that it includes boundary currents where the inertial terms play a role and so the Sverdrup constraint on the depth integrated flow is broken. Friction, eddy transfer, and wind forcing are treated as high-order effects that serve only to remove the indeterminacy of purely inviscid flow. Inertial boundary currents become an integral part of the solution—there is no need to match a Sverdrup interior to a western boundary region or, indeed, to restrict attention to midocean gyres where the western boundary plays no role. It should be noted, however, that here no account is taken of ventilation of the thermocline through surface outcrops, such as in the model of Luyten et al. (1983). The penetration of the gyre below the directly wind-driven layer is possible because, as in Rhines and Young's theory, the q contours are closed and do not connect to the lateral boundaries or outcrop at the surface.

In essence, a series of Fofonoff gyres is obtained "stacked" one on top of the other, but diminishing in strength and progressively receding to the latitude of the zero wind-stress curl with depth. We show that these simple, analytic solutions have many properties that compare remarkably well both with the results from a numerical model (the eddy-resolving quasi-geostrophic model described in Holland et al., 1984) and with observations of the subtropical recirculation (reported by McCartney, 1982). In particular, it is a fundamental property of our solutions that the isopycnals slope down toward the northern boundary of the subtropical gyre, in agreement both with the models and observations.

In section 2 the general methodology of our approach is set out and placed in context, reviewing the main results of barotropic free inertial theory. In section 3 free solutions in 1½ and 2½ layer models are considered and in section 4 an N½ layer model is described.

2. Forcing and dissipation and the q/ψ relationship

We shall consider flows governed by the potential vorticity equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{F} - \mathcal{D} \tag{2.1}$$

where q is the quasi-geostrophic potential vorticity used in several forms below, ψ the quasi-geostrophic streamfunction

$$J(\psi, q) = \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x},$$

the Jacobian of ψ and q, $\mathcal{F}$ a potential vorticity source, $\mathcal{D}$ a sink, x east, y north and t time.

Extending the barotropic studies of Fofonoff (1954) and Niiler (1966), we consider the constraints that must be satisfied by flows in steady ($\partial/\partial t \equiv 0$) baroclinic circulation. The most important of these constraints is that if the flow is free (i.e., $\mathcal{F} = \mathcal{D} = 0$) the potential vorticity q is conserved along a streamline ψ. The exact coincidence of q and ψ is, of course, an extreme asymptotic limit. It is of interest because it directly addresses the most important consequence of nonlinearity, that is, the modification of the potential vorticity by the motion field tending to align the ψ and q contours. Linear theory, at the other extreme, puts the emphasis on the departure of the ψ from a fixed q geometry.

The following conceptual model of the ocean circulation is adopted: it is supposed that the ocean circulation can achieve all the transfers necessary to offset potential vorticity sources and sinks by making only small adjustments to configurations in which q is constant along streamlines.

Then, writing

$$\mathcal{F} - \mathcal{D} = \epsilon G; \quad G \sim O(\beta|\nabla\psi|) \tag{2.2}$$

where β is the planetary vorticity gradient, df/dy, with f the Coriolis parameter; the fundamentally free nature of the circulation is reflected in the choice of a small ϵ. As is shown in appendix A, ϵ can be regarded as a recirculation index that can be expressed as the ratio of the Sverdrup velocity scale to the interior flow speed of a Fofonoff gyre. We expand the solution to Eq. (2.1) in terms of ϵ, writing

$$\left.\begin{aligned} \psi &= \psi_0 + \epsilon\psi_1 + \cdots \\ q &= q_0 + \epsilon q_1 + \cdots \\ G &= G_0 + \epsilon G_1 + \cdots \end{aligned}\right\}. \tag{2.3}$$

Direct substitution into Eq. (2.1) gives at zero order

$$J(\psi_0, q_0) = 0$$

$$\text{that is,} \quad q_0 = q_0(\psi_0). \tag{2.4}$$

Thus the zero-order problem allows an infinite number of possible choices for $q_0 = q_0(\psi_0)$. However, not all yield a valid solution at next order. In other words, not all $q_0(\psi_0)$ are compatible with the integral forcing and dissipation balances which appear at next order.

In an enclosed ocean basin the streamlines are closed, and integration of Eq. (2.1) over the region inside such a streamline of constant ψ_0 gives

$$\iint_{\text{inside } \psi_0} \nabla\cdot(\mathbf{v}q)dA = \iint_{\text{inside } \psi_0} \epsilon G dA \Rightarrow \oint (\mathbf{v}_0 + \epsilon\mathbf{v}_1 + \cdots)$$

$$\times (q_0 + \epsilon q_1 + \cdots)\cdot \mathbf{n}_0 dl = \epsilon \iint (G_0 + \epsilon G_1) dA$$

where $\mathbf{n}_0$ is a unit vector normal to closed ψ_0 contours.

Clearly all terms involving $\mathbf{v}_0$ in the line integral vanish, and by continuity so must the $\mathbf{v}_1 q_0$. Thus the $O(\epsilon)$ balance yields

$$\iint G_0 dA = 0, \tag{2.5}$$

giving a condition on the zero-order flow in Eq. (2.4). The physical content of Eq. (2.5) is straightforward, for it merely states that a necessary condition for steady inviscid flow to be only weakly perturbed by potential vorticity sources and sinks is that net forcing must balance net dissipation over a closed streamline of this flow. It is important to realize that Eq. (2.5) is a statement of integral balance; it does not imply a local balance between sources and sinks. An alternative derivation of the condition Eq. (2.5) along with an illuminating discussion can be found in Pierrehumbert and Malguzzi (1984).

The study of Niiler (1966) involved an application of the above balance to barotropic flow in a closed basin. He considered which $\mathcal{F}$ and $\mathcal{D}$ could equilibrate the inertial flows found by Fofonoff (1954).

Taking

$$\mathcal{F} = \frac{\mathbf{k}\cdot\text{curl}\,\boldsymbol{\tau}}{\rho H}$$

as the potential vorticity source provided by the wind-stress curl, $\mathbf{k}\cdot\text{curl}\,\boldsymbol{\tau}$, where $\mathbf{k}$ is a unit vector pointing vertically, ρ the density and H the depth, and

$$\mathcal{D} = \epsilon\nabla^2\psi_0,$$

a potential vorticity sink provided by bottom friction, the integral balance Eq. (2.5) takes on the form

$$\frac{1}{\rho H}\oint_{\psi_0} \boldsymbol{\tau}\cdot d\mathbf{l}=\epsilon\oint \mathbf{v}_0\cdot d\mathbf{l}. \quad (2.6)$$

Thus, over each closed streamline, the wind-stress forcing must be balanced by bottom friction. This is a constraint on the allowable functional relationships between q_0 and ψ_0.

It appears that no direct application of the balance, Eq. (2.6), to determine the flow field from a given wind-stress field is straightforward. It requires a numerical, iterative approach (cf. Merkine et al., 1985), which is probably at least as complicated as spinning up an ocean model to a steady state. Instead, Niiler (1966) adopted a Fofonoff gyre (with a linear q_0/ψ_0 relationship) as the zero-order flow satisfying

$$q_0=\nabla^2\psi_0+\beta y \quad (2.7a)$$

$$q_0=q_0(\psi_0) \quad (2.7b)$$

with

$$\frac{dq_0}{d\psi_0}=-\frac{\beta}{U_I}>0 \text{ a positive constant} \quad (2.7c)$$

where $U_I<0$ is the interior westward flow (see Fig. 1). He introduced a weak bottom friction and then, from Eq. (2.6), computed the form of the wind-stress curl forcing required to offset it. Precise details are not of interest. The important point is that a plausible wind-stress curl can always be chosen to equilibrate a barotropic, Fofonoff gyre in the face of weak bottom friction so long as [from (2.6)] $\oint \boldsymbol{\tau}\cdot d\mathbf{l}$ has the same sign as $\oint \mathbf{v}_0\cdot d\mathbf{l}$; the sense of circulation reflects the sign of the vorticity input. The detailed form of τ is not important.

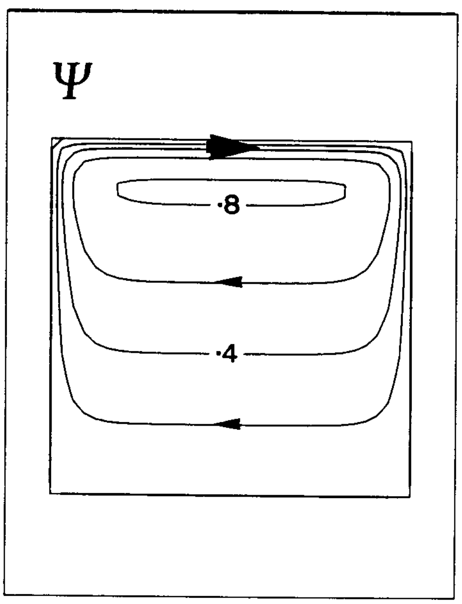

FIG. 1. Contours of ψ in a Fofonoff (1954) gyre with a uniform interior westward flow U_I. The boundary currents have a width $(U_I/\beta)^{1/2}$; in units of $|U_I|L$ with a contour interval (CI) of 0.2

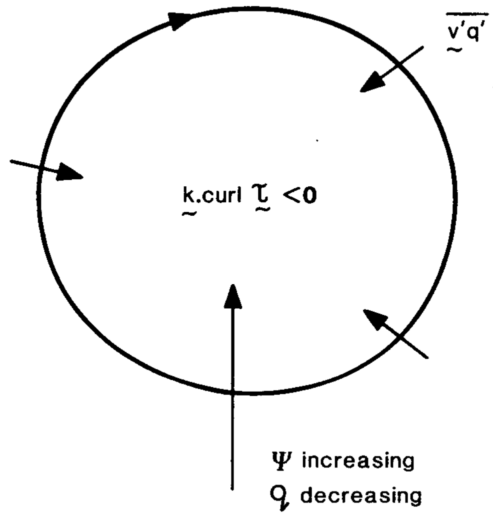

FIG. 2. An anticyclonic wind-stress curl generates anticyclonic motion, with ψ increasing into the gyre. If eddy q flux divergences are to balance forcing over a closed streamline then q must decrease into the gyre if the eddies transfer q downgradient. Thus $dq/d\psi$ must be negative.

Numerical experiments (e.g., Veronis, 1966) in which wind-stress curl is balanced solely by bottom friction provide confirmation, showing inertial behavior for reasonable choices of wind-stress curl.

Suppose, instead, that it is hypothesised that potential vorticity sources are balanced by a transfer of potential vorticity across mean streamlines due to a geostrophic eddy field [i.e., $\mathcal{D}=\nabla\cdot(\overline{\mathbf{v}'q'})$]. Such lateral down-gradient eddy transfer cannot (Niiler, 1966; McWilliams, 1977) equilibrate wind forcing in a barotropic Fofonoff gyre. Consider a region over which $\mathbf{k}\cdot\text{curl}\,\boldsymbol{\tau}<0$ (Fig. 2), generating anticyclonic motion; then in order that sources balance sinks, $\oint \overline{\mathbf{v}'q'}\cdot\mathbf{n}d\mathbf{l}$ must be negative, i.e., there must be an eddy flux of potential vorticity into the gyre to offset the interior sink of potential vorticity. Given an eddy flux $\overline{\mathbf{v}'q'}$, the divergent part of which is, in general, directed downgradient (see, for example, Marshall and Shutts, 1981), this is only possible if q_0 decreases into the gyre, i.e., $dq_0/d\psi_0<0$. Representing the divergent part of this flux by $-k\nabla q_0$, Eq. (2.5) takes on the form (noting that $\nabla q_0=(dq_0/d\psi_0)\nabla\psi_0$):

$$\frac{1}{\rho H}\oint \boldsymbol{\tau}\cdot d\mathbf{l}=-\frac{dq_0}{d\psi_0}\oint k\mathbf{v}_0\cdot d\mathbf{l}. \quad (2.8)$$

It is necessary that $dq_0/d\psi_0<0$ since the circulation must have the same sense as the wind-stress curl. A Fofonoff barotropic gyre cannot now be equilibrated since, with $dq_0/d\psi_0$ negative, inertial boundary layers cannot be supported; this can be seen most easily by supposing $dq_0/d\psi_0$ a negative constant in Eq. (2.7), which can therefore only support oscillatory solutions.

These considerations explain the qualitative differences between the steady state numerical results obtained by Bryan (1963) and those obtained by Veronis (1966). The introduction of a lateral friction into the Veronis (1966) model destroys his strongly inertial Fofonoff-type gyres, to be replaced by the standing Rossby-wave train described analytically by Moore (1963) and demonstrated numerically by Bryan (1963).

It is important to realize that the result $dq_0/d\psi_0 < 0$ follows from the hypothesis that geostrophic eddies flux potential vorticity down the mean q gradient. However, there is strong theoretical support for this conjecture. Systematic effects of eddies on mean flows are usually (and one is tempted to say always) associated with irreversible deformation of instantaneous q contours. Once this is accepted, the existence of an enstrophy sink is inevitable and thus the divergent potential vorticity flux must have a component that is directed downgradient (for a fuller discussion of this point, see Marshall, 1984). Thus, unlike McWilliams (1977) it is preferred not to contemplate the case where k can be negative. It should not be concluded, however, that this necessarily implies that almost free solutions of the type discussed here are unlikely to be relevant to actual oceanic flows. The problem lies more in the inability of the barotropic formulation to represent the dynamics of a baroclinic ocean.

We now go on to demonstrate that almost free solutions, in particular Fofonoff gyres, can be equilibrated by lateral transfer of q, provided that dynamical effects associated with vortex stretching are considered. Furthermore, these solutions have many realistic features.

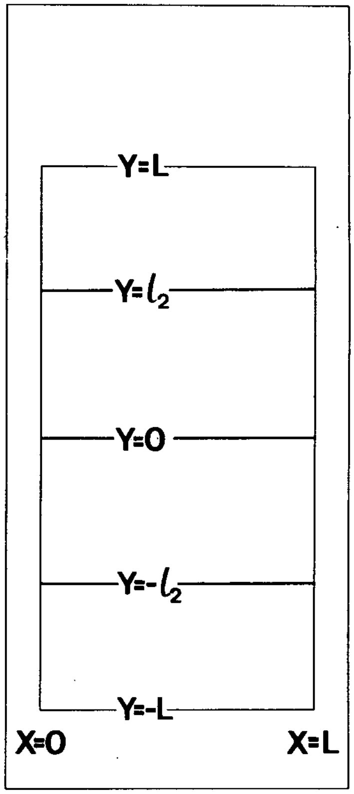

FIG. 3. The geometry of the basin. The latitudes $y = \pm l_2$ mark the north–south extent of the circulation in the second layer of the 2½-layer model.

3. Fofonoff gyres in 1½- and 2½-layer models

A two-gyre system (subpolar and subtropical) will be considered, driven by an antisymmetric wind-stress curl that imparts an equivalent amount of cyclonic vorticity into the subpolar gyre as it takes out from the subtropical gyre. The boundaries of the model ocean are assumed to be rectangular and to extend between $y = -L$ in the south to $y = L$ in the north and from $x = 0$ in the west to $x = L$ in the east (x positive eastward, y positive northward). (See Fig. 3.)

By symmetry, then, the latitude of the zero wind-stress curl line $y = 0$ marks the partition between the gyres. To emphasise how the role played by eddies in the lateral redistribution of potential vorticity q can be accommodated within the context of free-inertial circulation theory, we shall assume that vorticity balance is maintained entirely by an eddy flux of q between the gyres. A numerical model in which the instability of the internal jet plays this role is described in Marshall (1984). Thus at $y = 0$ it is supposed that the eastward-flowing jet is dynamically unstable. This instability provides the vorticity transfer required to offset the vorticity forcing, cyclonic to the north, anticyclonic to the south. Although they can be straightforwardly accommodated into our theory, frictional sinks of q will not be considered.

a. A 1½-layer model

We first consider the simplest possible extension of barotropic inertial theory and, instead of the absolute vorticity defined in Eq. (2.7a), adopt an "equivalent barotropic" formulation:

$$q = \nabla^2\psi + \beta y - F\psi \tag{3.1}$$

where $F = L_\rho^{-2}$; $L_\rho = (g'H/f_0^2)^{1/2}$ is the Rossby radius and g' the reduced gravity. The term $-F\psi$ is a crude (linearized) representation of the dynamical effects of vortex stretching associated with changes in layer depth.

The (statistically) steady state is a solution of the equation

$$J(\psi, q) = \mathcal{F} - \nabla \cdot (\overline{\mathbf{v}'q'}) \tag{3.2}$$

where $\mathcal{F}$ is the wind-stress curl and $\mathbf{v}'q'$ are eddy fluxes of q. The eddy flux divergence of q will be parameterized thus

$$\nabla \cdot (\overline{\mathbf{v}'q'}) = -\nabla \cdot (k\nabla q) \tag{3.3}$$

on the hypothesis that geostrophic eddies transfer potential vorticity down-gradient. The eddy transfer coefficient k is positive but, in general, a function of position.

Supposing that the flow is in an almost free configuration, we look for solutions of the form

$$\nabla^2\psi + \beta y - F\psi = q(\psi) \tag{3.4a}$$

and assume that

$$q(\psi) = C_1\psi + q_{10} \tag{3.4b}$$

where C_1 and q_1 are to be determined. To simplify the notation the subscript (0) indicating "zero order" has been dropped in Eq. (3.4) and in all subsequent occurrences. Since, over each closed streamline, wind-stress curl forcing is to be balanced by lateral eddy transfer of q, then from Eq. (2.8)

$$\frac{dq}{d\psi} = C_1 < 0 \tag{3.5}$$

i.e., C_1 must be negative. It will be assumed in what follows that C_1 is a negative constant so that the solutions we find are a generalization of Fofonoff's solution. Under the assumption of constant eddy transfer coefficient k made in appendix B, the assumed linear relationship between q and ψ is only strictly valid for a very specific form of the wind-stress curl forcing. However, arguments from statistical fluid mechanics involving the minimization of enstrophy (Bretherton and Haidvogel, 1976) or the maximization of entropy (Shutts, 1981, or Holloway, 1986) suggest that large-scale flows that arise spontaneously out of chaotic initial conditions may favor linear q/ψ relationships. We anticipate, then, that our solutions have relevance when the wind-stress curl is more realistic, a point that will be borne out by comparison with the numerical experiments of Holland et al. (1984).

Since by symmetry $y = 0$ is the partition between the gyres, the boundary conditions for Eq. (3.4) are

$$\left.\begin{array}{l}\psi = 0 \text{ along the boundary of each gyre} \\ x = 0, L \\ y = -L, 0, L\end{array}\right\}. \tag{3.6}$$

The constant q_{10} is determined by continuity of q across the northern and southern edges of the two-gyre system. This requires

$$q_{10} = \begin{cases}\beta L \text{ for the subpolar gyre } 0 < y < L \\ -\beta L \text{ for the subtropical gyre } -L < y < 0.\end{cases} \tag{3.7}$$

This choice leads to a discontinuity in q along the zero wind-stress curl line separating the two gyres. This seems appropriate since in response to opposite signed vorticity forcing, the counter-rotating gyres advect their q contours and concentrate then into a sharp gradient at $y = 0$. This discontinuity is a feature of the observed potential vorticity field in the upper midthermocline of the North Atlantic and particularly the North Pacific; see, for example, the escarpment in q marking the boundary between the subpolar and subtropical gyres in Fig. 11 of Keffer (1985). It is also present in the surface layer of numerical models; see Fig. 7 of Holland et al. (1984). On this sharp front in the q field, geostrophic eddies are spawned, fluxing q across and down the mean gradient and offsetting the potential vorticity source, cyclonic to the north, anticyclonic to the south.

Thus the problem to be solved is given by Eqs. (3.4), (3.5), (3.6) and (3.7). Solutions for the interior are easily written down, since here relative vorticity is negligible and so

$$\psi_I = \frac{\beta y - q_{10}}{C_1 + F}$$

giving

$$U_I = \frac{-\beta}{C_1 + F}. \tag{3.8}$$

Thus the interior flow will be to the west, supporting inertial boundary layers of width $(|U_I|/\beta)^{1/2}$ provided now that only

$$C_1 + F > 0.$$

The flow, plotted in Fig. 4, for the case $C_1 = -F/2$, has some interesting properties. The inclusion of a representation of vortex stretching enables the interior potential vorticity gradient to be reversed so that

$$\frac{\partial q}{\partial y} = \frac{dq}{d\psi}\frac{\partial\psi}{\partial y} = |U_I|\frac{dq}{d\psi} < 0$$

i.e., q decreases moving northward into the subtropical gyre, as it must if the potential vorticity source is to be balanced by lateral transfer of q. In baroclinic models this reversal of the q gradient south of the eastward-

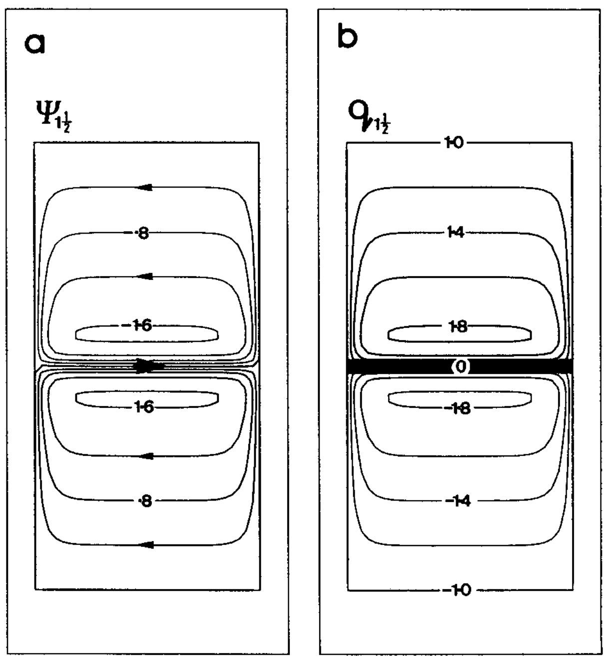

FIG. 4. The 1½-layer solutions plotted from Eqs. (3.4b) and (3.8) with inertial boundary layers included, for the case $C_1 = -F/2$: (a) streamfunction ψ (CI = 0.4) in units of $\beta L_\rho^2 L$; (b) potential vorticity q (CI = 0.2) in units of βL.

flowing internal jet makes the recirculation particularly susceptible to baroclinic instability.

However, one might speculate that the integral balance Eq. (2.8) is, in fact, a disjoint balance, with contributions from the wind-stress curl dominating in the interior, but lateral eddy fluxes dominating in the (unstable) internal jet at $y = 0$. In the present model then, the q/ψ relationship of the gyre could be set by eddy processes in the boundary currents and jets rather than (as in Rhines and Young, 1982) interior instability.

If the eddies are very efficient (k large) or the potential vorticity sources weak (τ small) then Eq. (2.8) implies that

$$C_1 = \frac{dq}{d\psi} \approx 0$$

i.e., q does not change across ψ contours, and the solution chooses the uniform q state. This, of course, is a natural end-state if the potential vorticity sources are weak and geostrophic eddies systematically erode potential vorticity gradients. In this limit the intensity of the gyre is independent of sources and sinks [$C_1 = 0$ in Eq. (3.8)] and given by

$$|U_I| \sim \beta L_\rho^2.$$

For a Rossby radius of $L_\rho \sim 30$ km this gives a not unreasonable value for the interior velocity of 2 cm s^{-1}, for $\beta \sim 2 \times 10^{-11}$ m^{-1} s^{-1}.

Although our 1½ layer solution can be no more than suggestive, it does have contact points with the ocean and particularly with the mean flows of eddy-resolving ocean circulation simulations. The upper-layer mean flow of the Holland et al. (1984) quasi-geostrophic model is strongly inertial, with eastward flow confined to a narrow jet and broad, gentle return flows. Moreover, $dq/d\psi < 0$ here, with reversed q gradients in the interior where vortex stretching overrides the planetary vorticity gradient. Our simple model suggests that $dq/d\psi$ is negative in this surface layer in order that potential vorticity sources can be balanced by the lateral transfer of q achieved by the geostrophic eddy field. This is certainly the dynamical balance at work in these weakly dissipative, highly nonlinear, thermodynamically inactive eddy-resolving models. To what extent this limit is an appropriate description of the ocean must await further detailed diagnostic and modeling studies. Examination of potential vorticity maps of the North Atlantic, however, does show strong advective control of the q contours with $dq/d\psi < 0$ in the near surface layers (see Fig. 4 of McDowell et al., 1982). This is consistent with our deduction from the circulation integral Eq. (2.8).

b. A 2½-layer model

The ideas of the previous section are now extended to "Fofonoff-like" gyres in a 2½-layer model in which two active layers of differing densities overlie a deep abyssal layer. For convenience it is assumed that both active layers have the same mean depth H and that the density jumps $\Delta\rho$ between each successive layer (including the abyssal layer) are equal. The quasi-geostrophic potential vorticity q for each of the two active layers ($n = 1$ for the upper layer, $n = 2$ for lower) is then given by

$$q_1 = \nabla^2\psi_1 + \beta y + F(\psi_2 - \psi_1) \tag{3.9a}$$

$$q_2 = \nabla^2\psi_2 + \beta y + F(\psi_1 - 2\psi_2) \tag{3.9b}$$

where ψ_n is the quasi-geostrophic streamfunction, $F = f_0^2/g'H$, $g' = g\Delta\rho/\rho_0$ is the reduced gravity, and ρ_0 a representative density.

The governing equations for the two active layers are then

$$J(\psi_1, q_1) = \mathcal{F} - \nabla \cdot (\overline{\mathbf{v}_1' q_1'}) \tag{3.10a}$$

$$J(\psi_2, q_2) = -\nabla \cdot (\overline{\mathbf{v}_2' q_2'}) \tag{3.10b}$$

where $\mathcal{F}$ is the curl of the wind stress (which acts only on the upper layer) and $\overline{\mathbf{v}'q'}$ are the eddy fluxes of q. As before, these are parameterized thus

$$\nabla \cdot (\overline{\mathbf{v}_n' q_n'}) = -\nabla \cdot (k_n \nabla q_n) \tag{3.11}$$

where the k_n are positive eddy transfer coefficients.

As before we shall assume that

$$q_1 = C_1\psi_1 + q_{10} \tag{3.12a}$$

$$q_2 = C_2\psi_2 + q_{20}, \tag{3.12b}$$

with the C_1 and C_2 determined by integral constraints of the form Eq. (2.8). In this case, they are obtained by integrating over the area enclosed by closed streamlines in each layer.

It is not difficult to see how flow in the upper layer can, through vortex stretching, modify the q geometry in layer 2, allowing flow around closed q contours. (This is discussed at length by Rhines and Young, 1982, where the depth integrated flow is constrained to be the Sverdrup transport.) Consider the case where the flow in layer 2 is small (or zero). With $\psi_2 = 0$, the equation for ψ_1 reduces to Eq. (3.4) of the 1½-layer model, with the previously considered solution Eq. (3.8). The q_2 is thus given by

$$q_2 = \beta y + F\psi_1 \tag{3.13}$$

using Eq. (3.9b) and is plotted in Fig. 5. We see that in the interior q_2 reaches the value zero at $y = -l_2$, say (in the subtropical gyre). For $y > -l_2$ there are closed q_2 contours. For $y < -l_2$, on the contrary, the q_2 contours are open and intersect the boundary. The $q_2 = 0$ contour of the subtropical gyre threads its way northward in the inertial boundary layers to meet the side boundaries at $y = 0$, along which line it closes itself. Outside this contour there can only be weak flow in the lower layer since all q_2 contours intersect the side boundaries. Within this contour, however, weak forcing

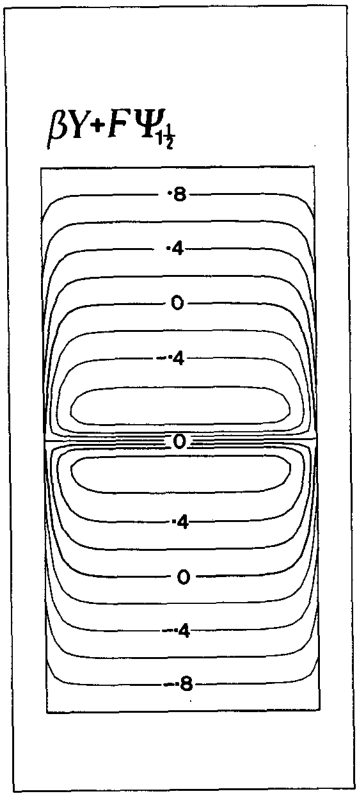

FIG. 5. The q_2 contours in the second layer of the 2½-layer model if $\psi_2 = 0$: $q_2 = \beta y + F\psi_{1½}$ with ψ_1 given by Eq. (3.8) with $C_1 = 0$ and inertial boundary layers appended; CI = 0.2. Within the $q_2 = 0$ contour, the q_2 contours are closed; outside the $q_2 = 0$ contour, they intersect the lateral boundary.

is capable of generating strong flow, as in the model of Rhines and Young (1982).

Thus, in the lower layer there can only be flow for $|y| < l_2$, where the q_2 contours are closed. If closed q_2 contours exist, then, because there are no potential vorticity sources, Eq. (2.8) takes the form

$$0 = -\frac{dq_2}{d\psi_2}\oint k_2 \mathbf{v}_2 \cdot d\mathbf{l}$$

which tells us that $dq_2/d\psi_2 = 0$, i.e., q_2 is uniform throughout the gyre.

Since, unlike in the upper layer, there are no interior potential vorticity sources capable of maintaining a discontinuity in q_2 at $y = 0$, then by symmetry q_2 must take on a uniform value across both gyres, i.e., $q_2 = 0$. Flow will be confined to this region. This is the "expulsion" of the q_2 contours to the north and south, which is spectacularly demonstrated in the eddy-resolving calculations of Holland et al. (1984). It is also a feature of the observed potential vorticity field at middepths in the main thermocline, where the contrast of q between the gyres disappears. See, in particular, Fig. 14 of Keffer (1984), which shows that the entire Northern Pacific has a homogeneous value of q stretching from 25°N right up to 65°N at densities (σ_θ = 27.0–27.3).

In the regions $|y| > l_2$, where the q_2 contours intersect the boundary, $\psi_2 = 0$ and Eq. (3.13) defines q_2, giving, using Eq. (3.8),

$$q_2 = \left(\frac{C_1 + 2F}{C_1 + F}\right)\beta y - \frac{F q_{10}}{(C_1 + F)} \quad (3.14)$$

i.e., the q_2 contours expelled from the uniform q_2 region increase the q gradients above the ambient gradient β.

Thus, in summary, the problem has been reduced to solving the following equations for ψ_n. In the region $|y| < l_2$:

$$\nabla^2\psi_1 + \beta y + F(\psi_2 - \psi_1) = C_1\psi_1 + q_{10} \quad (3.15a)$$

$$\nabla^2\psi_2 + \beta y + F(\psi_1 - 2\psi_2) = 0 \quad (3.15b)$$

and in the region $|y| > l_2$:

$$\nabla^2\psi_1 + \beta y - F\psi_1 = C_1\psi_1 + q_{10} \quad (3.16)$$

with

$$\left\{ q_{10} = \beta L \quad \begin{matrix} \text{for} & y > 0 \\ \text{for} & y < 0. \end{matrix} \right.$$

The latitude l_2 is determined by requiring that ψ_1 be continuous at $y = \pm l_2$. This is equivalent to calculating the "bowl" within which the circulation takes place, as will become clear in the N½-layer model described in section 4.

The problem Eqs. (3.15), (3.16) with boundary conditions (3.6) can be solved rather easily, since in the interior the relative vorticity $\nabla^2\psi_n$ is negligible. Thus, immediately from Eq. (3.15), for $|y| < l_2$;

$$\psi_1 = \frac{1}{(F + 2C_1)}[3\beta y - 2q_{10}] \quad (3.17a)$$

$$\psi_2 = \frac{1}{F(F + 2C_1)}[\beta y(2F + C_1) - F q_{10}]. \quad (3.17b)$$

The associated (zonal) velocities are

$$U_1 = -\frac{\partial\psi_1}{\partial y} = -3\beta/(F + 2C_1) \quad (3.18a)$$

$$U_2 = -\frac{\partial\psi_2}{\partial y} = -\beta(2F + C_1)/[F(F + 2C_1)]. \quad (3.18b)$$

It is clear that the flow is westward if $F > -2C_1$ (the condition that enables the flow to be closed by inertial boundary layers). Since C_1 is negative, the flow is stronger in the upper of the two active layers. In the limit that the eddies are efficient enough to homogenize the upper-layer potential vorticity (i.e., $C_1 = 0$), $U_1 = -3\beta/F$ and $U_2 = -2\beta/F$. Recalling that $F = L_\rho^{-2}$, we see that the upper-layer flow has amplitude $3\beta L_\rho^2$. It is clear from (3.18) that for C_1 different from 0, but negative, this amplitude will increase. In appendix A it is shown that a requirement for the validity of our theory is that this velocity be "large" compared with the Sverdrup velocity.

The above solution is not valid over the whole of the gyre but only in a latitude band $-l_2 < y < l_2$. Outside this band (that is, in the region $|y| > l_2$), $\psi_2 = 0$ and the upper-layer solution is obtained from

$$\beta y - F\psi_1 = C_1\psi_1 + q_{10}, \tag{3.19}$$

the equation which governs the flow in the 1½-layer formulation of the problem. The solution is given by Eq. (3.8).

The problem of calculating the latitude l_2 is equivalent to that of finding the bowl containing the circulation referred to earlier. Since we require ψ_1 to be continuous across $y = \pm l_2$, l_2 is found by substituting for ψ_1 from (3.19) into (3.15a) with $\psi_2 = 0$ and solving for y. This gives

$$l_2 = \frac{FL}{(C_1 + 2F)}. \tag{3.20}$$

This shows that when $C_1 = 0$, the gyre circulation in the lower layer extends only half as far from the zero wind-stress curl line ($y = 0$) as does the gyre circulation in the upper layer. This retreat toward the zero wind-stress curl line with depth will also be apparent in the N½-layer model considered in the next section.

To complete the solution, the structure of the boundary layers that close the flow are now examined. Our discussion is limited to the eastward jet along $y = 0$, as the structure of the eastern and western boundary currents is complicated by the need to match solutions across the $q = 0$ contour. Splitting ψ_i into interior ψ_{iI} and boundary layer parts ψ_{iB} [where ψ_{iI} is the solution we have already found given by (3.17)] the ψ_{iB} satisfy

$$\nabla^2\psi_{1B} + F(\psi_{2B} - \psi_{1B}) = C_1\psi_{1B} \tag{3.21a}$$

$$\nabla^2\psi_{2B} + F(\psi_{1B} - 2\psi_{2B}) = 0. \tag{3.21b}$$

The boundary condition Eq. (3.6) implies that $\psi_{nB} = -\psi_{nI}$ (which is known) along $y = 0$. Furthermore, the condition Eq. (3.7) on q_{10} gives $\nabla^2\psi_{1B} = \pm\beta L$ at $y = 0$, and the symmetry condition $q_2 = 0$ gives $\nabla^2\psi_{2B} = 0$ at $y = 0$. The solution to Eq. (3.21) consists of exponentially decaying terms with decay scale $1/d$ satisfying

$$d^4 - (C_1 + 3F)d^2 + F(2C_1 + F) = 0. \tag{3.22}$$

This has real solutions (to ensure exponential decay and hence boundary-layer structure) provided $F + 2C_1 > 0$, which is the same condition obtained previously to ensure westward flow in the interior [see (3.18)]. Choosing the Rossby radius $L_\rho = 30$ km, $L = 1000$ km, and $C_1 = 0$, solving Eq. (3.21) with the conditions on ψ_{nB} and $\nabla^2\psi_{nB}$, detailed above, give the results for the total $\psi_n = \psi_{nI} + \psi_{nB}$ plotted in Fig. 6.

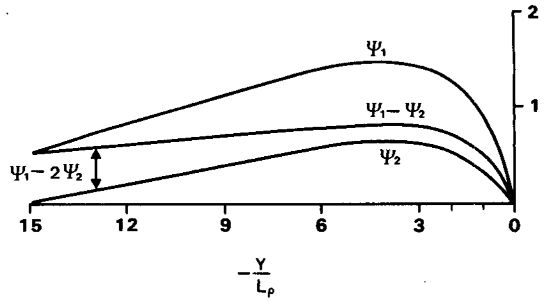

FIG. 6. The structure of the recirculation south of $y = 0$ in the 2½-layer model obtained by solving Eq. (3.22) for the case $L_\rho = 30$ km and $C_1 = 0$. Both upper and lower layers have inertial jets of width $\sim\sqrt{3}L_\rho$.

It may be seen that both upper and lower layers possess inertial jets of similar widths $\sim\sqrt{3}L_\rho$, in agreement with the theoretical $(|U_I|/\beta)^{1/2}$ scaling if $|U_I|$ is taken as $3\beta L_\rho^2$, the interior speed of the upper layer over the recirculation region $|y| < l_2$. Thus, although the interior velocities of the upper and lower layers lie in the ratio of 3:2, the greater extent of the upper-layer gyre implies velocities in the return eastward-flowing jet more nearly in the ratio of 2:1. Even with 2½ layers we see the beginnings of the tendency for a more barotropic flow in the interior with a more baroclinic eastward-flowing jet, as noted by Holland et al. (1984).

Unlike the 1½-layer model, flow in the upper layer now suffers vortex stretching. On a streamline corresponding to a given ψ_1, the downward displacement of the interface $\psi_1 - \psi_2$ is less in the eastward-flowing jet (because ψ_2 is greater here) than in the interior. Parcels of fluid in the upper layer are thus compressed in the western boundary current and stretched in the eastern boundary current.

Plan views of the solutions in the two active layers are plotted in Fig. 7 for the case $C_1 = 0$. We see counter-rotating inertial gyres in each of the two layers, with the lower gyres confined to a region of uniform q_2 of restricted meridional extent. In the 2½-layer model discussed here there is a discontinuity in velocity in the upper and lower layers along the latitudes $y = \pm l_2$, which mark the extent of the gyre circulation in the lower layer. There is thus a crude representation of a strong recirculation flanking the interior jet.

It should be noted that because of our choice of q_{10} [cf. Eq. (3.16)], q_1 has a discontinuity along the zero wind-stress curl line. Thus, with $C_1 = 0$, q is constant within each gyre but changes discontinuously across $y = 0$. This is a fundamental feature of our solution for it is here that we imagine eddy activity takes place, enabling the two-gyre system to be equilibrated.

It is interesting that in the lower layer there is no signature of the boundary currents in the potential vorticity field; there are fronts in the velocity and temperature fields but not in the q_2 field. This is a striking feature of the numerical solutions of Holland et al.

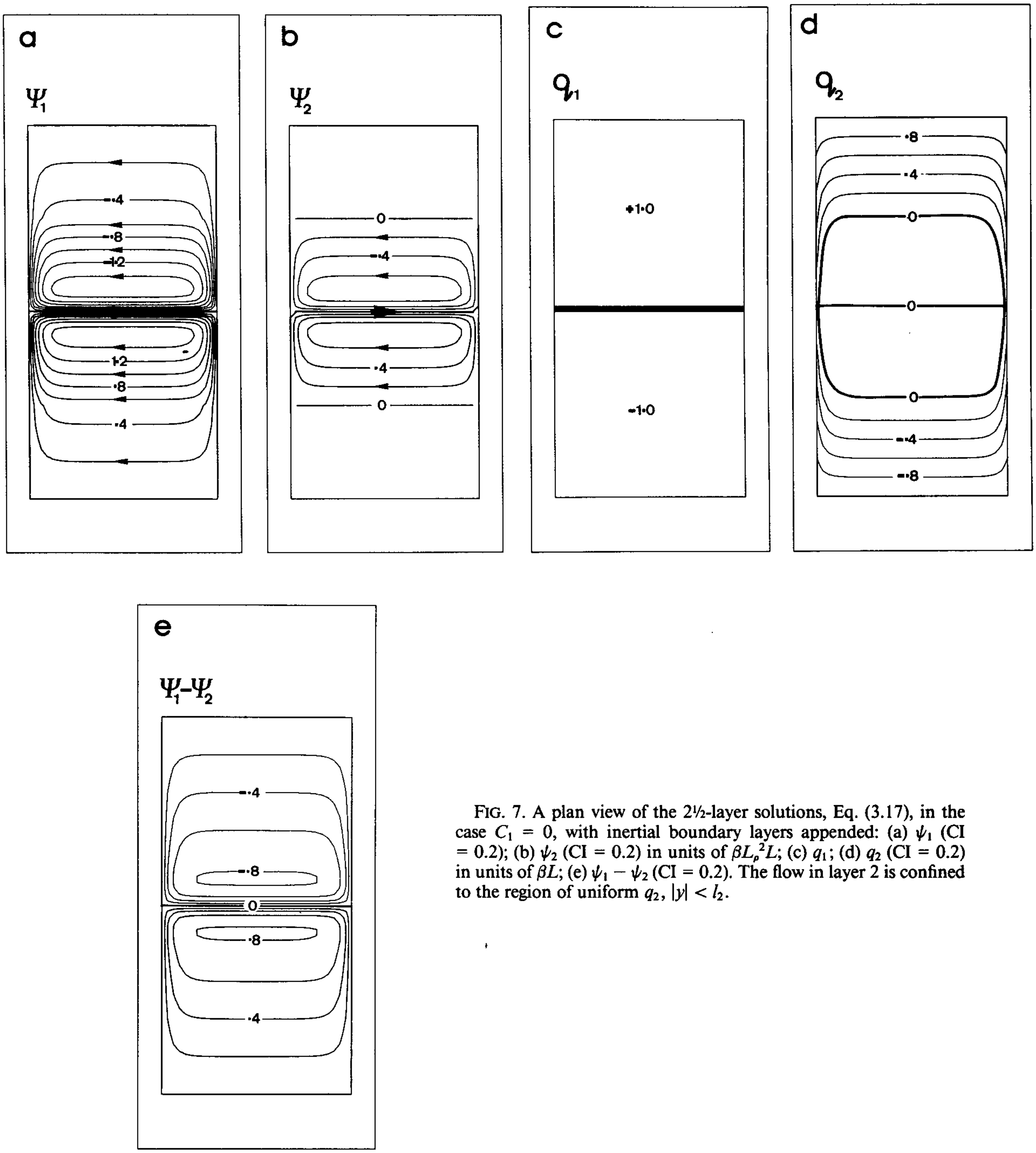

FIG. 7. A plan view of the 2½-layer solutions, Eq. (3.17), in the case $C_1 = 0$, with inertial boundary layers appended: (a) ψ_1 (CI = 0.2); (b) ψ_2 (CI = 0.2) in units of $\beta L_\rho{}^2 L$; (c) q_1; (d) q_2 (CI = 0.2) in units of βL; (e) $\psi_1 - \psi_2$ (CI = 0.2). The flow in layer 2 is confined to the region of uniform q_2, $|y| < l_2$.

(1984). (See in particular the mean ψ and q fields in Fig. 7 of that paper, which show pools of uniform q whose meridional extent progressively shrinks to the line of zero wind-stress curl on moving to deeper layers.) Moreover, beneath the surface layer the gyres equilibrate to the same uniform value of q.

Further confirmation of these gross features of the potential vorticity field is provided by the diagnostic study using the Gulf Stream '60 hydrographic survey presented in Bower et al. (1985). In the near-surface layers the Gulf Stream is seen to act as a barrier marking the boundary between high q water to the north and low q water to the south. Deeper down, tracers (O_2 and q) are uniform across the gyre boundary.

4. An N½-layer model

As a straightforward extension of the 2½-layer model of section 3 with two active layers, we now consider N

active layers overlying a motionless absyssal layer to study the vertical structure of the circulation.

We recall that in the 2½-layer model, variations in the thickness of the surface layer give rise to regions of closed q contours in the layer immediately beneath within which eddy processes homogenize q. Similarly, in an N½-layer model, vertical displacements of the interface between the $(n-1)$st and nth layers close off q contours in the nth layer. Thus, if as before, it is assumed that where q contours close in subsurface layers, q equilibrates to a uniform value across both subtropical and subpolar gyres, i.e., $q_n = 0$, then the following picture emerges.

Figure 8 shows a north–south section across the subtropical gyre, which will be the region considered here (the subpolar gyre is similar but with the interfaces sloping upward). The lines $y = -l_n$ are the latitudes at which q_n reaches zero, its value at the zero wind-stress curl line; these mark the southern boundary of the region of homogeneous q within which there is circulation. Thus, for example, the downward displacement of the interface between the second and third layers commences at $y = -l_2$ and so the line $y = -l_3$, along which squashing of the third layer raises the value of q_3 to zero, is displaced northward of $y = -l_2$. Similar arguments hold in the deeper layers. The region of homogeneous q thus recedes ever further northward moving to deeper layers.

The southern bound of the circulation and the velocities in each of the layers can easily be deduced. Let us consider again the case where the density differences between the layers and the layer thicknesses are identical. For simplicity it is supposed that the potential vorticity in the surface layer is constant; the shape of the interface between the surface layer and the layer beneath is therefore determined, as is the characteristic shear $(\Delta U)_1 = -\beta/F$, the velocity difference between the first and second layers. The velocities in subsurface layers can be calculated in terms of this characteristic shear; thus if q_2 is to be constant this implies an interface tilt between layers 2 and 3 *twice* that of the interface tilt between layers 1 and 2. The shear across the second interface is therefore $-2\beta/F$. Hence, where the second layer is the lowest layer in motion it has a velocity $-2\beta/F$ and the surface layer $-3\beta/F$. As shown in Fig. 8, this process continues as we proceed downward and poleward. If there are N active layers there are $N-1$ discontinuities in the surface layer across which the velocity increases discontinuously, moving poleward to the zero wind-stress curl line.

The latitudes $y = -l_n$ can be found as follows. The potential vorticity gradient in the nth layer south of the latitude $y = -l_n$, where there is no flow, is

$$\frac{\partial q_n}{\partial y} = \beta - F(\Delta U)_n = n\beta$$

since $(\Delta U)_n$ is the velocity difference across the $(n-1)$st and nth layers given by

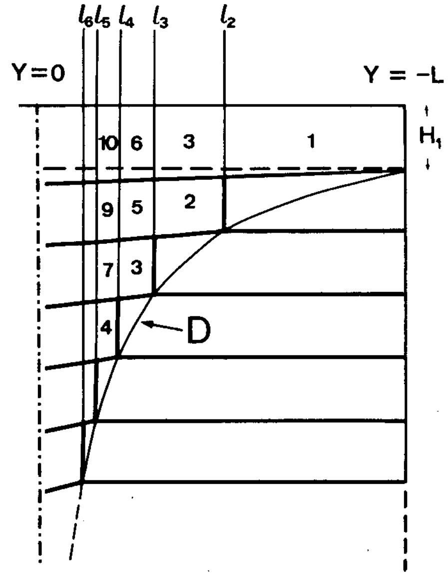

FIG. 8. A north–south section across the subtropical gyre showing the vertical structure of the circulation in the N½-layer model. The latitudes l_n mark the southern extent of the circulation in the layer n, and the numbers indicate the flow speed (westward) in units of βL_p^2. The curve D is the "bowl" within which the circulation is confined, given by Eq. (4.3). In the vicinity of $y = 0$ we expect an eastward-flowing jet where the isopycnals begin to slope upward from south to north.

$$(\Delta U)_n = (n-1)(\Delta U)_1 = -(n-1)\frac{\beta}{F}.$$

Thus

$$q_n = -\beta l_{n-1} + n\beta(y + l_{n-1})$$

for $-l_{n-1} < y < -l_n$ and

$$q_n = \beta y$$

for $y < -l_{n-1}$.

The latitude l_n is determined by noting that $q_n = 0$:

$$l_n = \frac{(n-1)}{n} l_{n-1} \quad \text{with} \quad l_1 = L$$

giving

$$l_n = \frac{L}{n}. \tag{4.1}$$

This is the generalization of the 2½-layer result Eq. (3.20) when $C_1 = 0$.

The depth of the bowl within which the circulation is confined, $D_n = nH_1$ is given by

$$D_n = \frac{LH_1}{l_n} \tag{4.2}$$

using Eq. (4.1).

The physics which leads to this striking result is very simple. Over the subtropical gyre the thickness of the surface layer increases linearly moving northward in order that f/h can remain constant and equal to the value of q_1 at the southern edge of the subtropical gyre:

$$\Delta h_1 = \frac{\beta(y+L)H_1}{f_0} \quad \text{for} \quad -L<y<0$$

where Δh_1 is the deviation of the upper layer depth about its undisturbed thickness H_1. Thus Δh_1 is always positive: vortex tubes become stretched in the surface layer as the zero wind-stress curl line is approached. However, the net vortex stretching over a column extending from $z = -D$ to the surface must sum to zero, and it is this that determines the depth of penetration of the circulation. In subsurface layers that are in motion, q_n must remain constant and equal to the value of q at $y = 0$, requiring that

$$\Delta h_n = \frac{\beta y H_n}{f_0} \quad \text{for} \quad -l_n<y<0.$$

Thus Δh_n is always negative; this compression (relative to the equilibrium layer thickness) of each subsurface layer increases moving southward.

So the requirement that the total vortex stretching be zero

$$\sum \Delta h_n = 0$$

implies that

$$H_1\beta(y+L) + \sum_{n=2}^{N} H_n\beta y = 0$$

giving

$$D = \sum_{n=1}^{N} H_n = \frac{-LH_1}{y} \quad \text{for} \quad -L<y<0. \tag{4.3}$$

This is an alternative derivation of the depth of penetration of the circulation deduced from a slightly different perspective. It is worthy of note that the above derivation is valid for layer thicknesses H_n not all equal; the result is completely independent of the stratification. It may be shown that, moving from a layered to a continuously stratified model, the shape of the bowl changes from a series of steps where Eq. (4.3) is valid only where the stepping occurs, to a smooth hyperbola where Eq. (4.3) holds everywhere.

This hyperbolic plunge cannot continue indefinitely, and it is natural to inquire as to the maximum depth of penetration of the bowl. The crucial parameter in our simplified model is the line $l_n = L/n$, which marks the southern boundary of homogenization in the nth layer. Consider a layer that is so deep that l_n lies inside the eastward-flowing inertial jet at $y = 0$ of width $(|U_I|/\beta)^{1/2}$. Here relative vorticity makes an important contribution toward maintaining a uniform value of q, allowing a thinning of the layers above and the eventual bowing upward of interfacial surfaces (cf. Fig. 6 in the 2½-layer case). The "burrowing" of the bowl, then, ceases in the nth layer where

$$\frac{L}{n} \approx \left(\frac{|U_I|}{\beta}\right)^{1/2}$$

Suppose that the width of the inertial jet is a function of the U_I attained at its eddy in the upper layer, given by

$$|U_I| = \frac{1}{2}n(n+1)\beta L_\rho^2,$$

then substitution into the above expression gives

$$n \sim \left(\frac{\sqrt{2}L}{L_\rho}\right)^{1/2}.$$

Taking $l_\rho = 30$ km and $L = 1000$ km, a crude estimate of the north–south extent of the recirculation gives $n \sim 6$. For a value of $H_1 \sim 200$ m this implies a maximum depth of penetration of ~1200 km. This compares well with the value of 1500 m usually quoted (e.g., Holland et al., 1984) for the depth of the circulation in the Pacific. The value predicted by the theory is, however, quite sensitive to our choice of L_ρ and L so the theory does not exclude the possibility of the circulation reaching the ocean floor, if less marked stratification and/or larger recirculations are supposed.

It is interesting to compare the vertical structure of our solution, Fig. 8, with the vertical structure of the recirculation revealed by the section (Fig. 9) crossing the subtropical gyre of the North Atlantic.

Figure 9 shows property sections along 50°W crossing the warm water subtropical gyre taken by *Atlantis* in 1956. Figure 9a is the potential density as a function of depth and Fig. 9b the potential vorticity as a function of potential density. The southern edge of the eastward-flowing Gulf Stream is marked by the vertical dotted line (for a detailed discussion, see McCartney, 1982).

The isopycnal surfaces dip down toward the southern edge of the stream at 37°N, turning sharply upward again at the latitude of the Gulf Stream. The region of flow clearly recedes rather rapidly to the axis of the Gulf Stream with depth. The bowl, although rather blurred, is nevertheless clearly evident and is not unlike our analytic construction (compare with Fig. 8).

The large isopycnal separation centered on $\sigma_\theta = 26.5$ is the 18° mode water of anomalously low potential vorticity. The shading in Figure 9b has been chosen to emphasise the low potential-vorticity water. The broad spatial distribution of the q field is in accord with that deduced from our circulation integrals: low q in the near-surface layers of the subtropical gyres, higher, but fairly uniformly distributed values of q beneath. It should be pointed out that the low potential vorticity waters in the near-surface layers of the subtropical North Atlantic are almost certainly formed by deep wintertime convection, resulting in regions of reduced stratification, which are thermohaline in origin. This reminds us that our model is almost diagnostic: once the q field has been set (either by the dynamics or thermodynamics) then, for given boundary conditions, the flow characteristics are determined. In the present case there are important potential vorticity sources due to diabatic processes that must be taken into account in

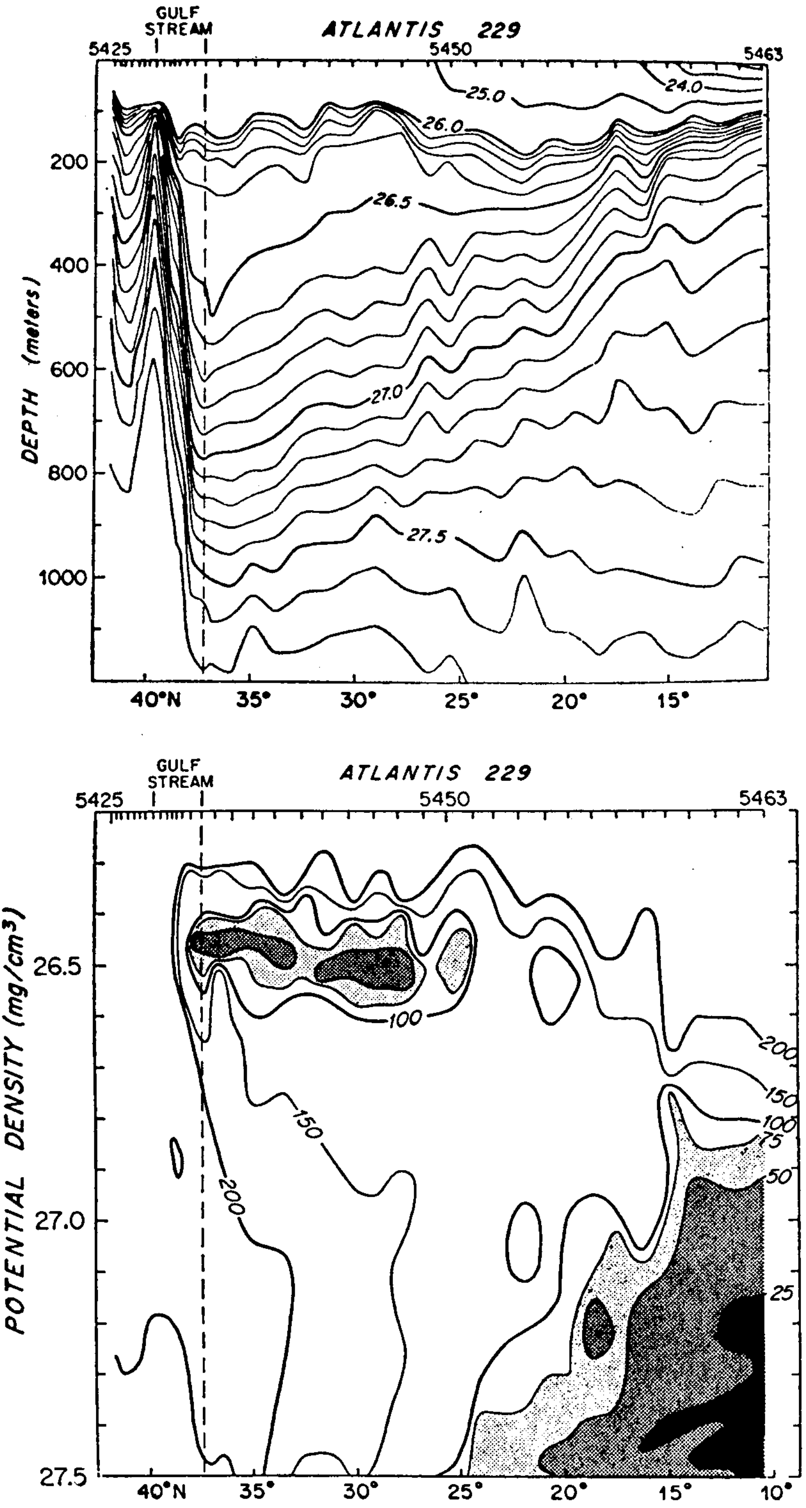

FIG. 9. Property sections for a section along 50°W made by *Atlantis* in 1956, between 13 (north) and 30 (south) November taken from McCartney (1982). The eastward flow of the Gulf Stream lies between stations 5432 (39°37′N) and 5439 (37°16′N). The vertical dashed line marks the southern edge of the Gulf Stream: station 5439, the dynamic height maximum. (a) Potential density σ_θ (in mg cm^{-3}), depth as ordinate): Eighteen Degree Water marked by larger isopycnal spacing centered at $\sigma_\theta = 26.5$ mg cm^{-3}. (b) Potential vorticity (in 10^{-14} cm^{-1} s^{-1}) with potential density as ordinate. Shading has been chosen to emphasize the low potential vorticity water masses: single, double and triple intensity denoting, respectively, less than 75, 50 and 25 × 10^{-14} cm^{-1} s^{-1}. The potential vorticity minimum layer centred near $\sigma_\theta = 26.5$ mg cm^{-3} represents the 18°C water.

addition to wind-stress curl forcing in any complete theory. These through (2.5), are also important in setting the q/ψ relationship of the recirculating waters of the subtropical gyre, where convection is an important source of low q waters.

The vertical structure of the circulation revealed in Fig. 8 also provides an interesting perspective from which to view the layer quasi-geostrophic numerical models. It would suggest that many layers are required at shallow depths to resolve the hyperbolic plunge of the bowl. The numerical model of Holland et al. (1984) has eight layers, most of them in the upper few hundred meters, which seem to resolve the rapid retreat of the circulation to the zero wind-stress curl line with depth fairly well. The model of section 3, of course, has only two layers and the rectangular hyperbola is poorly resolved as a discontinuous jump.

5. Discussion

In this study we have considered the consequences for the ocean circulation of supposing that quasi-geostrophic potential vorticity is conserved along streamlines of the flow. Following Niiler (1966) the indeterminacy of purely inviscid theory has been removed by a consideration of the necessary condition that potential vorticity sources balance sinks when integrated over a closed streamline.

From the hypothesis that potential vorticity sources are balanced by a lateral eddy transfer of potential vorticity (achieved by the geostrophic eddy field), it follows that in layers exposed to surface forcing $dq/d\psi < 0$, whereas beneath $dq/d\psi = 0$ with q constant.

Once these functional relationships are set, solutions for the zero-order free circulation can rather readily be found. They have a close affinity to Fofonoff's (1954) solutions and, indeed, are best regarded as generalizations of his barotropic solutions to baroclinic flow. A series of inertial gyres is obtained that progressively diminish in amplitude with depth and recede to the line of zero wind-stress curl with depth.

In addition to their academic interest we believe that there is evidence from modeling and, more to the point, observations that these solutions are very relevant to gyre-scale circulation in middle latitudes. This is because large fractions of the subtropical gyres are made up of recirculating waters in which parcels of fluid circulate many times before having their potential vorticity significantly changed. In other words, there is strong advective control of the q contours with close coincidence of the ψ and q contours.

The solutions we have found are, of course, inaccessible to linear theory. Linear theory puts the emphasis on the deviation of ψ from a prescribed but largely unknown q field. The other asymptotic limit considered here, $q = q(\psi)$, deserves equal attention. Indeed, it is possible that a more profitable approach would be to "linearize" about the present free mode, zero-order solutions rather than a state of rest ($q = \beta y$). The Sverdrup mass transport then appears at next order as a direct response to potential vorticity sources and sinks. This first order correction will be discussed in a subsequent paper.

In conclusion, it is hoped that this study will provide an interesting further example of how a focus on potential vorticity can bridge the gap between observations and theory. As more observations become available, particularly those that enable the relative vorticity contributions to q to be evaluated in boundary currents and jets, more quantitative tests of our theoretical speculations can be made.

APPENDIX A

A Consideration of the Scaling Parameter ϵ

Here we seek to justify the assertion, made in section 2, that the fundamentally free nature of a circulation is reflected in the choice of small ϵ in

$$J(\psi, q) = \epsilon G \tag{A1}$$

if G is

$$G \sim O(\beta|\nabla\psi|). \tag{A2}$$

Equations (A1) and (A2) imply that

$$\epsilon \sim O\left(\frac{|J(\psi, q)|}{\beta|\nabla\psi|}\right). \tag{A3}$$

The quantity ϵ can be interpreted in the following way. In steady flow a particle must move a distance ds in order that its potential vorticity be changed by an amount dq where

$$ds \sim \frac{|\nabla\psi|dq}{|J(\psi, q)|}.$$

If the potential vorticity gradient is dominated by β then this (meridional) distance is ds_β given by

$$ds_\beta \sim \frac{dq}{\beta}.$$

The ratio

$$\frac{ds_\beta}{ds} = \frac{|J(\psi, q)|}{\beta|\nabla\psi|} \tag{A4}$$

is the small parameter ϵ. Thus ϵ^{-1} is a recirculation index, a measure of the number of circuits of length $O(L)$ a fluid parcel must make in a nearly free flow before it loses an amount of vorticity βL (which in linear theory is lost in one pass through the frictional boundary current).

Given (A3), ϵ can be evaluated for a wind-driven barotropic Fofonoff gyre, where eddy flux divergences are unimportant and $J(\psi, q) = \mathbf{k} \cdot \text{curl}\, \tau/\rho H$. If the zonal velocity is u and meridional velocity v, then

$$\epsilon \sim \frac{|\mathbf{k}\cdot\text{curl}\,\tau|}{\rho\beta H(u^2+v^2)^{1/2}} \sim \frac{|v|}{(u^2+v^2)^{1/2}}$$

since $v = \mathbf{k}\cdot\text{curl}\,\tau/\rho H$ is the Sverdrup velocity. When $|v| \ll |u|$,

$$\epsilon \sim \frac{|v|}{|u|}$$

and Niiler's (1966) definition of ϵ is recovered as the ratio of the Sverdrup velocity scale to the zonal flow in the interior of a Fofonoff gyre.

APPENDIX B

The Determination of a Wind Stress Consistent with a Zero-Order Linear q–ψ Relationship in a Gyre Equilibrated by Downgradient Eddy Transfer of q

Our objective is to calculate a wind stress consistent with the linear q–ψ relationship ($q = C_1\psi - \beta L$; $C_1 < 0$) assumed at zero order in a Fofonoff gyre equilibrated by downgradient lateral eddy transfer of q. A 1½-layer quasi-geostrophic model is adopted for the subtropical part of the circulation shown in Fig. 3 ($-L < y < 0$; $0 < x < L$).

Given that the free component of the flow is of Fofonoff form (see Fig. 1), large potential vorticity gradients are confined to the boundary currents, so the major contributions from eddy fluxes in the integral balance (2.8), rewritten below for convenience,

$$\frac{1}{\rho H}\oint_{\psi_0} \tau\cdot d\mathbf{l} = -\frac{dq_0}{d\psi_0}\oint k\mathbf{v}_0\cdot d\mathbf{l} = |C_1|\oint k\mathbf{v}_0\cdot d\mathbf{l} \quad \text{(B1)}$$

can be assumed to occur in these boundary regions.

In order to evaluate the rhs of (B1) an expression for the velocity in the boundary currents must be found. It can be shown (following, for example, the Bernoulli integral approach of Niiler, 1966) that the northward velocity in the western boundary current is, dropping zeroes,

$$v(\psi_1\psi_I) = (F + C_1)^{1/2}(\psi_I - \psi) \quad \text{(B2)}$$

where ψ_i is the value which the streamfunction takes in the interior at a given latitude, and ψ is the streamfunction in the boundary current at the same latitude. Equation (B2) is also appropriate for speeds in northern and eastern boundary currents. The above expression is intuitively reasonable, since the interior v is negligible and the width of the boundary current is $(F + C_1)^{1/2}$.

The circulation integral (B1) can now be evaluated. The eddy flux across a streamline ψ in the western boundary current is

$$|C_1|k\int_{y_{\text{entry}}}^{0} v(\psi, y)dy$$

where y_{entry} is the latitude at which the streamline enters the boundary current and is equal to the latitude at which the streamline traverses the interior. Reexpression of the integral in terms of ψ yields

$$|C_1|k\int_{\psi}^{\psi_N} v(\psi, \psi_I)\frac{dy}{d\psi_I}d\psi_I$$

where ψ_N is the value of the streamfunction in the interior at the northern edge of the basin, $y = 0$. From Eq. (2.3)

$$\frac{dy}{d\psi_I} = \frac{F + C_1}{\beta},$$

and substituting for v from (B2), the integral takes the form

$$\beta^{-1}k|C_1|(C_1+F)^{3/2}\int_{\psi}^{\psi_N}(\psi_I - \psi)d\psi_I = \frac{1}{2\beta}k|C_1|(C_1+F)^{3/2}(\psi_N - \psi)^2.$$

By symmetry this is equal to the eddy flux into the gyre across the streamline in the eastern boundary current; the flux in the northern boundary current is given by

$$k|C_1|\int_0^L v(\psi, \psi_N)dx = k|C_1|L(C_1+F)^{1/2}(\psi_N - \psi).$$

Thus the total eddy flux into the gyre across a streamline ψ is

$$k|C_1|\{\beta^{-1}(C_1+F)^{3/2}(\psi_N - \psi)^2 + L(C_1+F)^{1/2}(\psi_N - \psi)\}.$$

Expressed as a function of y_{entry}, the latitude coincident with the streamline in the interior, we have (ρ and H are as in §2)

$$\frac{1}{\rho H}\oint \tau\cdot d\mathbf{l} = \frac{k\beta|C_1|}{(C_1+F)^{1/2}}\{y_{\text{entry}}^2 + Ly_{\text{entry}}\}. \quad \text{(B3)}$$

From (B3) a τ can be calculated. For simplicity, it is supposed that the wind stress is zonal and independent of x: $\tau = \{\tau(y), 0\}$. Then (B3) implies that a wind stress satisfying

$$\frac{1}{\rho\beta HL}\{\tau(0) - \tau(y)\} = \frac{k|C_1|}{(C_1+F)^{1/2}}\left(\frac{y^2}{L^2} + \frac{y}{L}\right) \quad \text{(B4)}$$

is indeed consistent with a linear q–ψ relationship.

Thus a wind-stress curl that varies linearly with y can equilibrate an equivalent barotropic Fofonoff gyre in the face of lateral eddy transfer of q. It has the same form as that obtained by Niiler (1966) for the barotropic Fofonoff gyre with bottom friction.

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