NOTES AND CORRESPONDENCE

Quasigeostrophic Potential Vorticity in Isentropic Coordinates

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ABSTRACT

An isentropic coordinate form of quasigeostrophic potential vorticity is derived. Compared with the well-known height and pressure coordinate versions, this isentropic form is more clearly related to Ertel's potential vorticity and the derivation from it is much simpler. A derivation from quasigeostrophic approximations to the governing equations is also given and the boundary conditions to be applied on isentropic surfaces are discussed. Analogous developments using isopycnal coordinates are given assuming an incompressible fluid model.

1. Introduction

In recent years there has been renewed meteorological interest in isentropic analysis. One of its attractions is that in isentropic coordinates there is no "vertical" motion if the flow is adiabatic. Furthermore, as atmospheric databases have improved, it has become feasible to diagnose isentropic distributions of the Ertel potential vorticity (herein referred to as the PV). The significance of this dynamical tracer was recently highlighted by Hoskins et al. (1985, herein referred to as HMR): the main attributes of PV are that it is materially conserved on isentropic surfaces in the case of adiabatic frictionless flow, and that given some balance condition on the wind field and appropriate boundary conditions it defines that field. Analogs of PV and its conservation are deemed desirable in any approximate formulation of fluid dynamical equations.

Quasigeostrophic (QG) models are among the most important approximate formulations in geophysical fluid dynamics. A key quantity in QG models is quasigeostrophic potential vorticity (QGPV) which takes the form $\mathcal{L}\psi$, where ψ is a streamfunction of the horizontal flow and $\mathcal{L}$ is a linear elliptic differential operator. Although some derivations (see Charney and Stern 1962; HMR) have emphasized the relationship of QGPV conservation in height or pressure coordinates to the conservation of PV on isentropic surfaces, the relationship between the two quantities themselves has often been obscured. This is because the development of QG models has usually been formulated in height or pressure coordinates (see Charney and Stern 1962; Phillips 1963; White 1977; Gill 1982; Bannon 1989).

Somewhat surprisingly, there have been few applications of QG theory in isentropic coordinates. Bleck (1973, 1974) has studied the performance of forecasting models that use isentropic coordinates in conjunction with the geostrophic approximation (see Charney and Phillips 1953); the resulting PV is nonlinear in the Montgomery streamfunction and hence not of the less accurate QGPV form. Gent and McWilliams (1984) have discussed a number of filtered models in isentropic coordinates but not the QG model; Hoskins and Draghici (1977) had previously examined the semigeostrophic equations in isentropic coordinates. Thermocline models of ocean gyres (see Welander 1971; Huang 1988) use isopycnal PV as a dependent variable but neglect the contribution of relative vorticity. In a study of stratified flow over isolated topography, Schär and Davies (1988) used an isentropic coordinate version of QGPV, which is similar to that given in (12) or (22) below, although of a slightly less general form. Our purpose here is to give a systematic development of QGPV in isentropic coordinates and

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to relate it to the more familiar height and pressure coordinate forms (which can be derived by simple transformation of the isentropic QGPV).

In section 2, an isentropic form of QGPV is derived by linearization and scale analysis of PV; note in particular the directness of the derivation as compared with that of QGPV in other coordinate systems and also the clear relationship between this form of QGPV and Ertel's PV. In section 3, the same QGPV is derived from QG forms of the governing equations written in isentropic coordinates. Boundary conditions are discussed in section 4. Section 5 deals with the analogous development of isopycnal QGPV when an incompressible fluid model is assumed.

2. Derivation from Ertel's potential vorticity

The hydrostatic approximation will be used, and we will assume adiabatic, frictionless conditions throughout. The PV can then be written in isentropic coordinates as

$$P = -g(f + \zeta_\theta)\left(\frac{\partial p}{\partial \theta}\right)^{-1}, \qquad (1)$$

where the potential temperature θ is given by

$$\ln(\theta) = \ln(T) - \kappa \ln\left(\frac{p}{p_s}\right) \qquad (2)$$

and g, κ, T, p, and p_s are, respectively, the acceleration due to gravity, the ratio of the gas constant (R), and the specific heat at constant pressure (C_p), the temperature, the pressure, and a reference surface pressure. The Coriolis parameter is denoted by f and ζ_θ is the relative vorticity of the horizontal motion evaluated on a θ surface. The Montgomery potential M is given by

$$M = gz + C_pT = gz + C_p\theta\left(\frac{p}{p_s}\right)^\kappa, \qquad (3)$$

where z is height, and its "vertical" derivative is given by

$$\frac{\partial M}{\partial \theta} = C_p(p/p_s)^\kappa = \frac{C_pT}{\theta}. \qquad (4)$$

Each atmospheric variable (e.g., p, T, M) can be expressed as the sum of a reference value (denoted by, subscript "0"), which is a function of θ only, and a deviation (denoted by "′"). For example,

$$p(x, y, \theta, t) = p_0(\theta) + p'(x, y, \theta, t), \qquad (5)$$

where (x, y) are orthogonal horizontal coordinates and t is time.

For small deviations from the reference state (i.e., when $p'/p_0 \ll 1$), (2) implies

$$\frac{T'}{T_0} \approx \kappa \frac{p'}{p_0}. \qquad (6)$$

From (4), using (6),

$$p' \approx \rho_0\theta \frac{\partial M'}{\partial \theta}, \qquad (7)$$

where ρ is density. We can then write

$$\left(\frac{\partial p}{\partial \theta}\right)^{-1} \approx \left(\frac{dp_0}{d\theta}\right)^{-1}\left\{1 - \frac{\partial p'/\partial \theta}{dp_0/d\theta}\right\}. \qquad (8)$$

On making the β-plane approximation, the geostrophic flow ($\mathbf{v}_g$) can be written

$$\mathbf{v}_g = \frac{1}{f_0}\mathbf{k} \times \nabla_\theta M' = \mathbf{k} \times \nabla_\theta \psi, \qquad (9)$$

where f_0 is a reference value of $f(= f_0 + \beta y)$, ∇_θ indicates that the gradient is evaluated on isentropic surfaces, and ψ is the geostrophic streamfunction. The relative isentropic vorticity in (1) may be replaced by its geostrophic value ($\zeta_{\theta g}$), given by

$$\zeta_{\theta g} = \nabla_\theta^2 \psi \qquad (10)$$

if the Rossby number is small. On using (8) and setting $f + \zeta_{\theta g}$ to f_0 when it multiplies $\partial p'/\partial \theta$, the PV can be written as

$$P \approx -\frac{g}{dp_0/d\theta}\left\{f_0 + \beta y + \nabla_\theta^2\psi - \frac{f_0^2}{(dp_0/d\theta)}\frac{\partial}{\partial \theta}\left(\rho_0\theta\frac{\partial \psi}{\partial \theta}\right)\right\}, \qquad (11)$$

and we identify the quantity Q, defined by

$$Q = \beta y + \nabla_\theta^2\psi - \frac{f_0^2}{(dp_0/d\theta)}\frac{\partial}{\partial \theta}\left(\rho_0\theta\frac{\partial \psi}{\partial \theta}\right) \qquad (12)$$

as the isentropic counterpart of the usual QGPV.

The scaling conditions for the approximations made in deriving (11) must be considered. Suppose that the motion has a horizontal velocity scale V, a horizontal space scale L, and a vertical θ scale $\Delta\theta$. From (9), $M' \sim f_0VL$. If the vertical space scale of the motion, H, is of order H_0 ($= RT_0/g$, the scale height associated with T_0), the approximation (8) is valid only if

$$\frac{\partial p'/\partial \theta}{dp_0/d\theta} \sim \frac{p'}{p_0} \sim \frac{\theta}{\Delta\theta}\frac{f_0VL}{gH_0} \ll 1. \qquad (13)$$

So the conditions for the validity of (11) are

$$\mathrm{Ro} \ll 1; \quad (\mathrm{Ri\ Ro})^{-1} \ll 1, \qquad (14)$$

where $\mathrm{Ri} = N^2H^2/V^2$ and $\mathrm{Ro} = V/(f_0L)$ are the Richardson and Rossby numbers, and the buoyancy frequency N is defined by $N^2H/g = \Delta\theta/\theta$. On the synoptic scale typical values of Ri and Ro are 10^2 and 10^{-1}, respectively, so the conditions (14) are reasonably well satisfied.

The second condition in (14) may be regarded as requiring the flow to be Boussinesq in the sense that fractional changes of p, T (and hence ρ) over an isentropic surface are small. It contrasts with the corresponding criterion in height or pressure coordinates. For example, in height coordinates $p' \sim \rho_0 f_0 VL$ for nearly geostrophic motion, and it follows that

$$\frac{p'}{p_0} \sim \frac{\rho'}{\rho_0} \sim \frac{f_0 VL}{gH} = \frac{N^2H}{g}(\mathrm{Ri\,Ro})^{-1} \qquad (15)$$

(see Charney 1963). Since $N^2H/g < 1$ ($N^2H_0/g = \kappa$ for an isothermal atmosphere), (15) gives a less stringent condition for Boussinesq behavior on constant height surfaces than does (14) for Boussinesq behavior on isentropic surfaces. In the height coordinate case condition, however, (14) must be obeyed if the contribution to Ertel's PV of the horizontal vorticity components multiplied by the horizontal θ gradients is to be negligible (Green 1970, p. 170). In both coordinate systems, therefore, (14) is the most demanding condition for the validity of the appropriate QGPV and is equivalent to requiring that the pressure on an isentropic surface does not deviate far from its reference value.

Of perhaps greater interest is the simplicity of the derivation of the isentropic coordinate form of QGPV as compared with the derivation of the corresponding forms in other coordinate systems. In height coordinates, for example, we must first argue that the terms in Ertel's PV that involve the horizontal components of vorticity are negligible (see above). Then the vertical advection of Ertel's PV must be allowed for by using a quasigeostrophic form of the thermodynamic equation to eliminate the vertical velocity. Neither of these steps is necessary in the isentropic coordinate derivation shown above, which is consequently much more direct. The absence of adiabatic vertical advection in θ coordinates also leads to a simple relationship between QGPV and Ertel's PV itself [see (11) and (12)]. In all coordinate systems other than θ (or functions thereof), this relationship is obscured; various authors identify the QGPV in height or pressure coordinates as a "pseudo PV" (HMR, p. 911; Charney (1971), p. 1089).

The p- or z-coordinate forms of QGPV may be derived from the isentropic form (12) by transforming the vertical coordinate and neglecting various terms that are small if conditions (14) are obeyed. In the p-coordinate case, for example, ∇_θ^2 may be replaced by ∇_p^2, $p_0(\theta)$ by p, and $\rho_0(\theta)\theta$ by $\rho_0(p)\theta_0(p)$, given (14). Thus,

$$\beta y + \nabla_\theta^2\psi - \frac{f_0^2}{(dp_0/d\theta)}\frac{\partial}{\partial\theta}\left(\rho_0\theta\frac{\partial\psi}{\partial\theta}\right) \approx \beta y + \nabla_p^2\psi + f_0^2\frac{\partial}{\partial p}\left(\frac{1}{N_p^2}\frac{\partial\psi}{\partial p}\right), \qquad (16)$$

where $N_p^2 = -(\rho_0\theta_0)^{-1}d\theta_0/dp$. The rhs of (16) is a familiar form of QGPV in p coordinates.

Given the simplicity of the derivation of the isentropic form of QGPV, it appears that the quickest way to derive QGPV conservation in p or z coordinates from conservation of Ertel's PV is to derive the θ-coordinate version and then to obtain the p- or z-coordinate form by transforming the vertical coordinate.

3. Derivation from the governing equations

Conservation of the quantity Q, as defined by (12), may be derived from quasigeostrophic forms of the horizontal momentum, continuity, hydrostatic, and state equations assuming adiabatic, frictionless conditions. The hydrostatic primitive equations in θ coordinates are then

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\theta\right)\mathbf{v} + f\mathbf{k}\times\mathbf{v} + \nabla_\theta M = 0 \qquad (17)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\theta\right)\frac{\partial p}{\partial\theta} + \frac{\partial p}{\partial\theta}\nabla_\theta\cdot\mathbf{v} = 0, \qquad (18)$$

together with (2) and (4). Corresponding quasigeostrophic forms (with the β-plane approximation) are

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\theta\right)\mathbf{v}_g + f_0\mathbf{k}\times\mathbf{v}_a + \beta y\mathbf{k}\times\mathbf{v}_g = 0 \qquad (19)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\theta\right)\frac{\partial p'}{\partial\theta} + \frac{dp_0}{d\theta}\nabla_\theta\cdot\mathbf{v}_a = 0 \qquad (20)$$

together with (6), (7), and (9). Here $\mathbf{v} = \mathbf{v}_g + \mathbf{v}_a$, and M, T, and p have been expanded as indicated by (5). These equations are valid approximations of the hydrostatic primitive forms if conditions (14) are satisfied.

From (19), it follows that

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\theta\right)(\beta y + \nabla_\theta^2\psi) + f_0\nabla_\theta\cdot\mathbf{v}_a = 0. \qquad (21)$$

Hence, using (7) and (20),

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\theta\right)\left\{\beta y + \nabla_\theta^2\psi + \frac{f_0^2}{\rho_0 g(dz_0/d\theta)}\frac{\partial}{\partial\theta}\left(\rho_0\theta\frac{\partial\psi}{\partial\theta}\right)\right\} = 0 \qquad (22)$$

in which the conserved quantity is the isentropic QGPV, Q, defined by (12). Schär and Davies (1988) give a less general form of (22) in which the factor $\rho_0\theta$ is replaced by a representative surface value.

The above derivation of (22) depends on the retention of the term $(\partial/\partial t + \mathbf{v}_g \cdot \nabla_\theta)(\partial p'/\partial\theta)$ in the QG continuity equation (20). The QG model in θ coordinates is thus essentially of the "modified" rather than the "standard" type (see White 1977). This resemblance is borne out by a consideration of the energetics of the system. By multiplying (22) by ψ, an energy equation can be obtained whose quadratic form is similar to that of equation (34) of White (1977).

4. Quasihorizontal boundary conditions

At rigid horizontal boundaries the condition to be applied is $w = Dz/Dt = 0$. If isentropes deviate from horizontal surfaces by much less than a scale height of the motion, which is guaranteed by satisfying (14), then it is an acceptable approximation to apply this boundary condition on an isentrope, θ_s, which is near the physical boundary.

Hence, we apply on $\theta = \theta_s$,

$$w = \left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\theta\right) z' = 0, \tag{23}$$

indicating that height deviations are simply advected around on the isentropic surface. Considering the bottom boundary, positive deviations would imply low values of θ at the ground, whereas negative deviations would imply high θ values there. Since

$$\frac{gz'}{f_0} = \psi - \theta \frac{\partial\psi}{\partial\theta}, \tag{24}$$

(23) may be written as

$$-\frac{1}{\theta}\frac{\partial\psi}{\partial t} + \left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\theta\right)\frac{\partial\psi}{\partial\theta} = 0. \tag{25}$$

This expression contains a "non-Doppler" term: $(\theta)^{-1}\partial\psi/\partial t$ (White 1982). It is negligible if $\Delta\theta/\theta \ll 1$, where $\Delta\theta$ is the "height" scale of the motion in terms of θ. Since $\Delta\theta/\theta = N^2H/g$, the condition for the neglect of the non-Doppler term is precisely the same as in the height coordinate case (White 1977).

If the non-Doppler term in (25) is neglected then the boundary condition is simply that $\partial\psi/\partial\theta$, which is a function of pressure deviation according to (7) and (9), is advected around on the bounding isentrope. A useful conceptual device can now be utilized (see Bretherton 1966). For example, the bottom boundary, θ_s, can be considered to be a pressure surface (where $\partial\psi/\partial\theta = 0$) if the quantity

$$\frac{f_0^2\theta}{g(dz_0/d\theta)} \left.\frac{\partial\psi}{\partial\theta}\right|_{s+} \delta(\theta - \theta_s) \tag{26}$$

is added to the rhs of (12). Here "$s+$" indicates evaluation just above the bottom boundary and δ is the Dirac delta function. The physical equivalence here is of interest in that (26) represents a discontinuity in the pressure just above the lower boundary. This implies the presence of a nonzero mass of air with very low static stability.

5. The incompressible equations

For its oceanographic relevance, QGPV is now formulated in isopycnal coordinates for the case of a heterogeneous incompressible fluid.

Hydrostatic isopycnal PV may be well approximated by

$$-\frac{(f + \zeta_\rho)}{\rho_s}\left(\frac{\partial z}{\partial\rho}\right)^{-1}, \tag{27}$$

where ζ_ρ is the relative vorticity on an isopycnal surface and ρ_s is a constant value of density. [In the ocean, the fractional density variations are of the order of 10^{-3}, so the use of ρ_s instead of ρ in (27) is amply justified.]

The Montgomery potential is defined as

$$M = gz + \frac{p}{\rho} \tag{28}$$

(see Starr 1945), and hydrostatic balance can be expressed as

$$\frac{\partial}{\partial\rho}(\rho M) = gz. \tag{29}$$

As before, if horizontal deviations of static stability are small compared to the reference values, then a binomial expansion of (27) using (29) leads to the QGPV quantity

$$\beta y + \nabla_\rho^2\psi - \frac{f_0^2}{g(dz_0/d\rho)}\frac{\partial^2}{\partial\rho^2}(\rho\psi), \tag{30}$$

where ∇_ρ^2 is the horizontal Laplacian evaluated on an isopycnal surface.

Again Ri Ro $\gg 1$ is the most stringent condition for validity [see (14)] and requires that the deviation depth of an isopycnal surface from its reference value be small compared with the vertical scale of the motion. On the scale of ocean gyres Ri $\sim 10^4$ and Ro $\sim 10^{-3}$; again, therefore, conditions (14) are reasonably well satisfied.

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