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A Note on Rotational and Divergent Eddy Fluxes

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ABSTRACT

If the deviation of mean flow from mean temperature contours is small, it is shown that a part of the eddy heat flux can be separated out which circulates around eddy potential energy contours, and has a component up/down the mean temperature gradient if there is flow advection of eddy potential energy into/out of the region. If the mean flow is strong, this rotational flux is large and results in regions of up- and downgradient flux. It is a prominent feature of maps of geostrophic eddy fluxes in the ocean and atmosphere.

1. Introduction

In baroclinically unstable flow, average eddy heat fluxes must have a net component down the mean gradient in order that mean available potential energy may be released. Similarly, quasi-geostrophic potential vorticity fluxes must have a net component down the mean potential vorticity gradient to offset dissipation of the eddy enstrophy (see Holland and Rhines, 1980; Rhines and Holland, 1979; Rhines, 1979). However, locally this association between mean gradient and eddy flux is not so strong (in regions where mean flow is large), because the sense of the eddy fluxes reflects not only the generation of eddies but also their decay downstream. Eddy fluxes in the atmosphere (Lau, 1978; Lau and Wallace, 1979, hereafter LW) and ocean (Holland and Rhines, 1980, hereafter HR) often have large rotational parts with both up- and downgradient components. Here it is shown that it is a rotational part of the eddy heat flux, circulating around eddy potential energy contours, that represents the spatial growth and decay of eddies, and balances the mean flow advection of eddy potential energy.

2. Up- and downgradient eddy fluxes

The steady-state eddy potential energy equation (see HR) relates the flux of heat across the mean temperature gradient $\overline{\mathbf{v}'T'}\cdot\nabla\bar{T}$, to the rate of conversion of eddy potential energy to eddy kinetic energy $\overline{w'T'}\partial\bar{T}/\partial z$, and the advection of eddy potential energy by the flow $\overline{\mathbf{v}\cdot\nabla T'^2/2}$. It can be written, neglecting sources and sinks of heat and the advection of eddy potential energy by the eddy velocity,

$$\bar{\mathbf{v}}\cdot\nabla\frac{\overline{T'^2}}{2}+\overline{\mathbf{v}'T'}\cdot\nabla\bar{T}+\overline{w'T'}\frac{\partial\bar{T}}{\partial z}=0, \qquad (1)$$

where $\mathbf{v}$ is the horizontal velocity, w the vertical velocity and T is the temperature (taken proportional to the density). The overbar represents a time-average long compared to an eddy life time, and prime the deviation from the average.

In baroclinically unstable regions, loss of eddy

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potential energy, by both its conversion into eddy kinetic energy $\overline{w'T'}\partial\bar{T}/\partial z > 0$ and its transport downstream, $\bar{\mathbf{v}}\cdot\nabla\overline{T'^2}/2 > 0$ is balanced by the downgradient transfer of heat with $\overline{\mathbf{v}'T'}\cdot\nabla\bar{T} < 0$. But in regions of eddy decay, where $\overline{w'T'}\partial\bar{T}/\partial z$ is small or even negative, the spatial decay of eddies $\bar{\mathbf{v}}\cdot\nabla\overline{T'^2}/2 < 0$ often results in upgradient transfer of heat with $\overline{\mathbf{v}'T'}\cdot\nabla\bar{T} > 0$.

In the case of potential vorticity, local dissipation by the enstrophy cascade can be balanced by both conversion from the mean field and flow advection. In decay regions, flow advection can dominate, resulting in upgradient potential vorticity fluxes.

Thus HR explained why, in much of the energetic upper layer of an ocean basin, the Austausch coefficient for temperature and potential vorticity can be negative if eddies, generated in boundary currents, decay away in the interior. HR also mention the role played by advection in eddy fluxes in the atmosphere: upper-level heat fluxes generated by transient weather systems are along isotherms, and even upgradient, over the western United States and Europe (see LW). In this note the advective contribution in the eddy potential energy equation is investigated further.

3. Rotational fluxes balancing flow advection

We define a reference mean state $\bar{\mathbf{v}}_0$, $\bar{T}_0$ as mean flow along mean temperature contours, and the deviation from this state (induced by heat sources and sinks, eddies and vertical motion) $\bar{\mathbf{v}}_1$, $\bar{T}_1$:

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_0 + \bar{\mathbf{v}}_1,$$
$$\bar{T} = \bar{T}_0 + \bar{T}_1,$$

so that

$$\left.\begin{aligned} \bar{\mathbf{v}}_0\cdot\nabla\bar{T}_0 = J(\bar{\psi}_0, \bar{T}_0) = 0 \\ \text{and therefore} \\ \bar{\psi}_0 = \bar{\psi}_0(\bar{T}_0) \end{aligned}\right\}. \qquad (2)$$

If $\bar{\mathbf{v}}_1 \ll \bar{\mathbf{v}}_0$, $\bar{T}_1 \ll \bar{T}_0$, then Eq. (1) may be approximated by

$$\bar{\mathbf{v}}_0\cdot\nabla\frac{\overline{T'^2}}{2} + \overline{\mathbf{v}'T'}\cdot\nabla\bar{T}_0 + \overline{w'T'}\frac{\partial\bar{T}}{\partial z} = 0,$$

where

$$\bar{\mathbf{v}}_0 = \mathbf{k}\wedge\nabla\bar{\psi}_0 = \frac{d\bar{\psi}_0}{d\bar{T}_0}\,\mathbf{k}\wedge\nabla\bar{T}_0$$

using (2). Noting that

$$(\mathbf{k}\wedge\nabla\bar{T}_0)\cdot\nabla\overline{T'^2} = -(\mathbf{k}\wedge\nabla\overline{T'^2})\cdot\nabla\bar{T}_0,$$

we may write the eddy potential energy equation in the form

$$\left.\begin{aligned} \overline{(\mathbf{v}'T')_D}\cdot\nabla\bar{T}_0 + \overline{w'T'}\frac{\partial\bar{T}}{\partial z} \\ \overline{(\mathbf{v}'T')_R}\cdot\nabla\bar{T}_0 + \bar{\mathbf{v}}_0\cdot\nabla\frac{\overline{T'^2}}{2} \end{aligned}\right\} = 0, \qquad (3)$$

where

$$\overline{(\mathbf{v}'T')_D} = \overline{\mathbf{v}'T'} - \overline{(\mathbf{v}'T')_R},$$
$$\overline{(\mathbf{v}'T')_R} = \frac{1}{2}\frac{d\bar{\psi}_0}{d\bar{T}_0}\,\mathbf{k}\wedge\nabla\overline{T'^2}.$$

So if the deviation of the mean flow from the mean temperature contours is small, the eddy heat flux separates naturally into two parts.

The $\overline{(\mathbf{v}'T')_R}$ flux is associated with the spatial growth and decay of eddies and balances the mean flow advection of $\overline{T'^2}$: it circulates around the $\overline{T'^2}$ contours and has a component up/down the mean temperature gradient if there is a transport of $\overline{T'^2}$ into/out of the region (see Fig. 1). The $\overline{(\mathbf{v}'T')_D}$ flux balances the conversion of eddy potential energy to kinetic energy: it has a component down the mean gradient if there is conversion of eddy potential energy to eddy kinetic energy.

If there is a linear relationship between $\bar{\psi}_0$ and $\bar{T}_0$, with $d\bar{\psi}_0/d\bar{T}_0$ a constant independent of horizontal position (the $\bar{\psi}_0$ flow is equivalent barotropic), then $\overline{(\mathbf{v}'T')_R}$ becomes a nondivergent, purely rotational heat flux, rotating around the $\overline{T'^2}$ contours. It is closely related to the rotational flux identified in LW. (LW assumed $\psi' \propto T'$ rather than $\bar{\psi} \propto \bar{T}$. The correspondence is exact if the total flow is equivalent barotropic.)

The extension to potential vorticity or relative vorticity fluxes is straightforward. This time we choose a $\bar{\psi}_0 = \bar{\psi}_0(\bar{q}_0)$ as a reference, where $\bar{q}_0$ is the mean quasi-geostrophic potential vorticity. If the deviation from this state is small, a component of the eddy potential vorticity flux $\overline{(\mathbf{v}'q')_R}$ can be separated out which balances the flow advection of eddy enstrophy in the eddy enstrophy equation.

$$\overline{(\mathbf{v}'q')_R} = \frac{1}{2}\frac{d\bar{\psi}_0}{d\bar{q}_0}\,\mathbf{k}\wedge\nabla\overline{q'^2}.$$

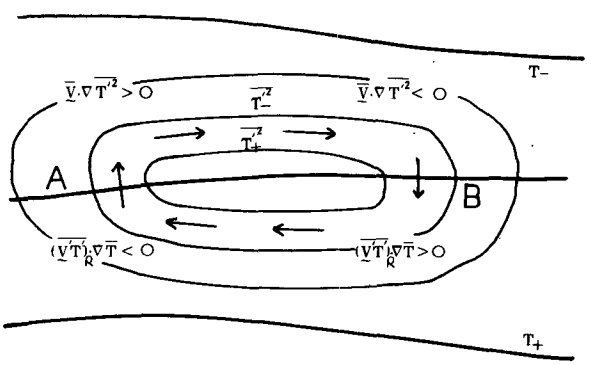

FIG. 1. Schematic picture showing rotational heat fluxes (arrows) in relation to the mean temperature (open contours) and the eddy potential energy (closed contours). Eddies generated near A grow, move downstream and decay near B. The rotational heat fluxes circulate anticyclonically around the eddy potential energy contours. Downgradient rotational fluxes balance the spatial growth of eddies near A and upgradient fluxes balance the spatial decay of eddies near B.

Again, if $d\bar{\psi}_0/d\bar{q}_0$ is a constant independent of horizontal position, the $(\overline{\mathbf{v}'q'})_R$ is a nondivergent vector flux. The remainder of the q flux is downgradient if there is local dissipation of eddy enstrophy.

a. Eddy fluxes in the ocean

The above analysis is particularly relevant to the eddy-resolving ocean model of Holland (1978). Here the temperature is set up mainly by the upper layer flow, $\bar{\psi}_1$, with weak eddy driven flow in the lower layer, $\bar{\psi}_3$ (compare Fig. 1a with 1c in HR). Because $\bar{\psi}_3 \ll \bar{\psi}_1$ then, to good approximation

$$J(\bar{\psi}_2, \bar{h}_2) \approx 0$$

in the sense that $\left|\frac{\partial\bar{\psi}_2}{\partial x}\frac{\partial\bar{h}_2}{\partial y}\right| \Big/ |J(\bar{\psi}_2, \bar{h}_2)| \gg 1$ and

$$\frac{d\bar{\psi}_2}{d\bar{h}_2} = \text{a known positive constant,}$$

where $\bar{\psi}_2$ is the stream function at the interface, and $\bar{h}_2$ is the height of the interface proportional to the temperature $\bar{\psi}_1 - \bar{\psi}_3$.

The purely rotational flux

$$(\overline{\mathbf{v}_2'h_2'})_R = \frac{1}{2}\frac{d\bar{\psi}_2}{d\bar{h}_2}\,\mathbf{k} \wedge \nabla\overline{h_2'^2}$$

is a dominant feature of the maps of $\overline{\mathbf{v}_2'h_2'}$ (Fig. 11, HR) rotating around contours of eddy potential energy P' (Fig. 4c, HR). For this reason the Austausch coefficient $A_T = -\overline{\mathbf{v}_2'h_2'}\cdot\nabla\bar{h}_2/|\nabla\bar{h}_2|^2$ (Fig. 12, HR) is a misleading indicator of eddy-mean flow interaction: a negative A_T does not necessarily imply that the eddies are sharpening up the mean gradients. The extensive negative regions are due largely to dynamically inert, nondivergent eddy fluxes, balancing the mean flow advection of $\overline{h_2'^2}$.

The remaining component of the eddy heat flux $(\overline{\mathbf{v}_2'h_2'})_D$ *does* represent conversion from the mean field, and its divergence can alter the mean temperature gradient. Maps of $(\overline{\mathbf{v}_2'h_2'})_D$ in relation to $\bar{h}_2$, or maps of $(A_T)_D = -(\overline{\mathbf{v}_2'h_2'})_D\cdot\nabla\bar{h}_2/|\nabla\bar{h}_2|^2$ would give a more reliable indication of the sense of the potential energy conversions.

In a continuously stratified ocean, beneath the surface Ekman layer and away from diffusive boundary currents, mean flow along $\bar{q}_0$ contours is a good approximation, provided the eddy-driven mean flow is weak. In the Holland model, though, the upper layer flow is driven across the $\bar{q}$ contours by the wind-stress curl and our assumption $\bar{\psi}_0 = \bar{\psi}_0(\bar{q}_0)$ is not valid. However, the association between mean flow advection and rotational fluxes is still evident. In the upper layer with strong mean flow the eddy q flux has a large rotational component (see Fig. 7d, HR)[1] and is directed up the mean gradient where there is transport of eddy enstrophy into the region. In the lower layer with weak mean flow, the rotational component has almost disappeared (see Fig. 7a, HR)[1] and the eddy q flux is directed down the mean gradient.

b. Eddy fluxes in the atmosphere

In an analysis of Northern Hemispheric synoptic charts LW finds that eddy fluxes are dominated by rotation above 500 mb (see Fig. 6, LW) and that only in the lower troposphere are the fluxes irrotational and directed down the mean temperature gradient (see Fig. 3, LW). If the eddy heat flux near the tropopause, however, is separated into rotational and irrotational fluxes, the rotational fluxes circulate around the storm tracks, with irrotational fluxes directed down the mean temperature gradient (see Fig. 7, LW).

These observations again support the idea that rotational heat fluxes are associated with flow advection of eddy potential energy. In the upper troposphere the rotational component is large and it rotates in the correct sense to balance the flow advection of $\overline{T'^2}$ by the jet stream. Lower down, where mean flow is weaker, the eddy heat flux, no longer dominated by rotation, points down the mean temperature gradient.

4. Discussion

Lau (1978) recognized that midlatitude eddy fluxes in the atmosphere have large rotational parts circulating around the storm tracks, as a consequence of the quasi-geostrophy of the eddy motion.

In this note we have pointed out that such rotational fluxes are also a feature of quasi-geostrophic fluxes in the ocean models. Further, we suggest that it is the rotational, nondivergent fluxes that largely balance the flow advection term in the eddy variance equation, implying that they are associated with the spatial growth and decay of eddies. The irrotational, divergent flux is left to balance conversion from the mean field. The dominance of the rotational component (particularly in regions of eddy decay) can mask dynamically important divergent fluxes which are interacting with the mean. Cross-gradient eddy heat fluxes must balance both conversion from the mean field and flow advection of eddy potential energy. Thus the local Austausch coefficient for heat contains information about the local conversion and transport of eddy potential energy. Similarly, the Austausch coefficient for potential vorticity contains information about the local dissipation and the transport of eddy enstrophy. This detracts from inferences drawn about eddy-mean flow interaction based on local measurements of the Austausch coefficient.

Geostrophic eddy parameterization schemes for use in low resolution atmosphere and ocean

[1] Fig. 7 of HR is labelled incorrectly: (a) to (c) are lower layer fluxes, (d) to (f) upper layer fluxes.

models, seek a closure for the eddy flux divergence of potential vorticity. Schemes based on the local downgradient transfer of potential vorticity (White and Green, 1981; Marshall, 1981) may not be so grossly in error (in strong, curved flow regimes) as previously suggested (Harrison, 1978; Holland and Rhines, 1980) if the transfer coefficients are reinterpreted as relating the divergent part of the flux to the mean gradient. It is intended to carry out calculations using the eddy statistics from an eddy resolving model, to test whether a diffusive parameterization for the irrotational, divergent potential vorticity flux is appropriate.

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A Weak Formulation of the Shallow-Water Equations for a Rotating Basin

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ABSTRACT

A weak formulation of the equation for the elevation field arising from the shallow-water equations for a rotating inviscid fluid has been developed.

The difficulties of this problem, which are due chiefly to the fact that the normal velocity along the contour is given by a linear combination of normal and tangential derivatives of the elevation field, are overcome. As a result, many problems, until now treated using the elevation and velocity component variables, might be solved dealing only with the elevation.

1. Introduction

It is well known that the linear shallow-water equations for a rotating basin, in the case of inviscid fluid and time-harmonic motion, can be reduced to a single equation in the elevation field. However, rarely has this equation been approached numerically, because of its boundary conditions which involve a linear combination of normal and tangential derivatives.

For example, if the depth is assumed to be constant, such an equation reduces to a normal Helmholtz equation for which a variational formulation exists. However, this formulation relates the elevation field to its normal derivative, that is no more proportional to the normal velocity field, as in the case in which the earth's rotation can be neglected. Hence, such a formulation is scarcely meaningful from a physical point of view and turns out to be of little help.

Now we will present a weak formulation of the problem which has the advantage of relating the elevation field to the normal velocity along the contour, and hence can be directly used in practical problems.

2. Basic equations

Let ζ be the elevation field, $\mathbf{u}$ the velocity, g the gravity acceleration, h the depth of the considered basin, f the local Coriolis parameter ($f = 2\Omega \sin\varphi$, where Ω is the angular speed of rotation of the earth and φ is the latitude) and ω, either real or complex, the angular velocity of the motion, assumed to depend on the time τ by the factor $e^{-\iota\omega\tau}$, where ι is the imaginary unit.

Then the linearized shallow-water equations read

$$-\iota\omega\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} + g\nabla\zeta = 0, \quad (1)$$

$$-\iota\omega\zeta + \nabla\cdot(h\mathbf{u}) = 0, \quad (2)$$

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